

ON THE CRITICAL CHOQUARD EQUATION WITH POTENTIAL WELL

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ABSTRACT. In this paper we are interested in the following nonlinear Choquard equation

$$-\Delta u + (\lambda V(x) - \beta)u = (|x|^{-\mu} * |u|^{2_\mu^*})|u|^{2_\mu^*-2}u \quad \text{in } \mathbb{R}^N,$$

where $\lambda, \beta \in \mathbb{R}^+$, $0 < \mu < N$, $N \geq 4$, $2_\mu^* = (2N - \mu)/(N - 2)$ is the upper critical exponent due to the Hardy-Littlewood-Sobolev inequality and the nonnegative potential function $V \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$ such that $\Omega := \text{int} V^{-1}(0)$ is a nonempty bounded set with smooth boundary. If $\beta > 0$ is a constant such that the operator $-\Delta + \lambda V(x) - \beta$ is non-degenerate, we prove the existence of ground state solutions which localize near the potential well $\text{int } V^{-1}(0)$ for λ large enough and also characterize the asymptotic behavior of the solutions as the parameter λ goes to infinity. Furthermore, for any $0 < \beta < \beta_1$, we are able to find the existence of multiple solutions by the Lusternik-Schnirelmann category theory, where β_1 is the first eigenvalue of $-\Delta$ on Ω with Dirichlet boundary condition.

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1. INTRODUCTION AND MAIN RESULTS

In this paper we are concerned with the existence of solutions of the Choquard type equation

$$(1.1) \quad -\Delta u + V(x)u = (|x|^{-\mu} * |u|^q)|u|^{q-2}u, \quad \text{in } \mathbb{R}^N,$$

where $N \geq 4$, $0 < \mu < N$ and $V(x)$ is the external potential. This type of equation goes back to the description of the quantum theory of a polaron at rest by S. Pekar in 1954 [35] and the modeling of an electron trapped in its own hole in 1976 in the work of P. Choquard, as a certain approximation to Hartree-Fock theory of one-component plasma [28]. In some particular cases, this equation is also known as the Schrödinger-Newton equation, which was introduced by Penrose in his discussion on the selfgravitational collapse of a quantum mechanical wave function [36].

In last decades, a great deal of mathematical efforts have been devoted to the study of existence, multiplicity and properties of solutions of the nonlinear Choquard equation (1.1). For constant potentials, if $N = 3$, $q = 2$ and $\mu = 1$, the existence of ground states of equation (1.1) was obtained in [28, 30] by variational methods. Involving the qualitative properties of the ground states, the uniqueness was proved in [28] and the nondegeneracy was established in [27, 40]. For equation (1.1) with general q and μ , the regularity, positivity, radial symmetry and decay property of the ground states were proved in [12, 31, 32]. Moreover, the existence of positive ground states under the assumptions of Berestycki-Lions type in [33]. For the existence of sign-changing solutions of the nonlinear Choquard equation, we refer the readers to the references [15, 21, 22]. For nonconstant potentials, if V is a continuous periodic function with $\inf_{\mathbb{R}^3} V(x) > 0$, noticing that the nonlocal

2010 *Mathematics Subject Classification.* 35J20, 35J60, 35A15.

Key words and phrases. Critical Choquard equation; Hardy-Littlewood-Sobolev inequality; Potential well; Lusternik-Schnirelmann category.

Zifei Shen and Fashun Gao were partially supported by NSFC (11671364);

* Minbo Yang is the corresponding author, he was partially supported by NSFC (11571317) and ZJNSF(LY15A010010).

term is invariant under translation, we can obtain easily the existence result by applying the Mountain Pass Theorem. If V changes sign and 0 lies in the gap of the spectrum of the Schrödinger operator $-\Delta + V$, the problem is strongly indefinite, and the existence of solution for $q = 2$ was considered in [11] and the existence of infinitely many geometrically distinct weak solutions in [1].

If the nonlinear Choquard equation is equipped with deepening potential well of the form $\lambda a(x) + 1$ where $a(x)$ is a nonnegative continuous function such that $\Omega = \text{int} (a^{-1}(0))$ is a non-empty bounded open set with smooth boundary. Moreover, suppose that Ω has k connected components, more precisely,

$$(1.2) \quad \Omega = \bigcup_{j=1}^k \Omega_j$$

with

$$(1.3) \quad \text{dist}(\Omega_i, \Omega_j) > 0 \quad \text{for } i \neq j,$$

the existence and multiplicity of multi-bump shaped solution in [5].

We need to point out that all the existing results for the nonlinear Choquard equation (1.1) require that the exponent q satisfies

$$\frac{2N - \mu}{N} < q < \frac{2N - \mu}{N - 2}.$$

To understand why the range of q make sense, it is necessary to recall the well-known Hardy-Littlewood-Sobolev inequality.

Proposition 1.1. (*Hardy-Littlewood-Sobolev inequality*). (See [29].) Let $t, r > 1$ and $0 < \mu < N$ with $1/t + \mu/N + 1/r = 2$, $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. There exists a sharp constant $C(t, N, \mu, r)$, independent of f, h , such that

$$(1.4) \quad \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x - y|^\mu} dx dy \leq C(t, N, \mu, r) |f|_t |h|_r,$$

where $|\cdot|_q$ for the $L^q(\mathbb{R}^N)$ -norm for $q \in [1, \infty]$. If $t = r = 2N/(2N - \mu)$, then

$$C(t, N, \mu, r) = C(N, \mu) = \pi^{\frac{\mu}{2}} \frac{\Gamma(\frac{N}{2} - \frac{\mu}{2})}{\Gamma(N - \frac{\mu}{2})} \left\{ \frac{\Gamma(\frac{N}{2})}{\Gamma(N)} \right\}^{-1 + \frac{\mu}{N}}.$$

In this case there is equality in (1.4) if and only if $f \equiv (\text{const.})h$ and

$$h(x) = A(\gamma^2 + |x - a|^2)^{-(2N - \mu)/2}$$

for some $A \in \mathbb{C}$, $0 \neq \gamma \in \mathbb{R}$ and $a \in \mathbb{R}^N$.

By the Hardy-Littlewood-Sobolev inequality, for every $u \in H^1(\mathbb{R}^N)$, the integral

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^q |u(y)|^q}{|x - y|^\mu} dx dy$$

is well defined if

$$\frac{2N - \mu}{N} \leq q \leq \frac{2N - \mu}{N - 2}.$$

Here $\frac{2N - \mu}{N}$ is the lower critical exponent and $2_\mu^* := \frac{2N - \mu}{N - 2}$ is the upper critical exponent due to the Hardy-Littlewood-Sobolev inequality. The critical problem for the Choquard equation is an interesting topic and has attracted a lot of attention recently. The lower critical exponent case was studied in [33], some existence and nonexistence results were established if the potential $1 - V$ should not decay to zero at infinity faster

than the inverse square of $|x|$. In order to study the critical nonlocal equation with upper critical exponent 2_μ^* , we use $S_{H,L}$ to denote the best constant defined by

$$(1.5) \quad S_{H,L} := \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2_\mu^*} |u(y)|^{2_\mu^*}}{|x-y|^\mu} dx dy \right)^{\frac{N-2}{2N-\mu}}}.$$

A critical Choquard type equation on a bounded domain of \mathbb{R}^N , $N \geq 3$ was investigated in [18, 19], there the authors generalized the well-known results obtained in [6, 10]. In [18] it was observed that

Proposition 1.2. (See [18].) *The constant $S_{H,L}$ defined in (1.5) is achieved if and only if*

$$u = C \left(\frac{b}{b^2 + |x-a|^2} \right)^{\frac{N-2}{2}},$$

where $C > 0$ is a fixed constant, $a \in \mathbb{R}^N$ and $b \in (0, \infty)$ are parameters. What's more,

$$S_{H,L} = \frac{S}{C(N, \mu)^{\frac{N-2}{2N-\mu}}},$$

where S is the best Sobolev constant.

Let $U(x) := \frac{[N(N-2)]^{\frac{N-2}{4}}}{(1+|x|^2)^{\frac{N-2}{2}}}$ be a minimizer for S , see [41] for example, then

$$(1.6) \quad \tilde{U}(x) = S^{\frac{(N-\mu)(2-N)}{4(N-\mu+2)}} C(N, \mu)^{\frac{2-N}{2(N-\mu+2)}} U(x)$$

is the unique minimizer for $S_{H,L}$ that satisfies

$$-\Delta u = \left(\int_{\mathbb{R}^N} \frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu} dy \right) |u|^{2_\mu^*-2} u \quad \text{in } \mathbb{R}^N$$

and

$$\int_{\mathbb{R}^N} |\nabla \tilde{U}|^2 dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\tilde{U}(x)|^{2_\mu^*} |\tilde{U}(y)|^{2_\mu^*}}{|x-y|^\mu} dx dy = S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}.$$

Moreover, for every open subset Ω of \mathbb{R}^N ,

$$(1.7) \quad S_{H,L}(\Omega) := \inf_{u \in D_0^{1,2}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2 dx}{\left(\int_{\Omega} \int_{\Omega} \frac{|u(x)|^{2_\mu^*} |u(y)|^{2_\mu^*}}{|x-y|^\mu} dx dy \right)^{\frac{N-2}{2N-\mu}}} = S_{H,L},$$

$S_{H,L}(\Omega)$ is never achieved except when $\Omega = \mathbb{R}^N$. That means, for bounded domain Ω there are no nontrivial solutions for

$$-\Delta u = \left(\int_{\Omega} \frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu} dy \right) |u|^{2_\mu^*-2} u \quad \text{in } \Omega.$$

On the other hand, similar to the observation made in [9], if $V(x) = \lambda$ is a constant and $q = \frac{2N-\mu}{N-2}$ in (1.1) while u is a classical solution, then we can establish the following Pohožaev identity

$$\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{\lambda N}{2} \int_{\mathbb{R}^N} |u|^2 dx = \frac{N-2}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2_\mu^*} |u(y)|^{2_\mu^*}}{|x-y|^\mu} dx dy,$$

thus we can obtain

$$\lambda \int_{\mathbb{R}^N} |u|^2 dx = 0,$$

which means that there are no nontrivial solutions with $\lambda \neq 0$. Hence it is quite interesting to know how the behavior of the potential function or the perturbation of the critical term will affect the existence of solutions for critical Choquard equation.

If the critical part was perturbed by a subcritical term, the existence of ground states was investigated in [4] there the authors also studied the semiclassical limit problem for the singularly perturbed Choquard equation in \mathbb{R}^3 and characterized the concentration behavior by variational methods. For the problem with sign-changing potential, a strongly indefinite Choquard equation with critical exponent was studied in [20] via generalized linking theorem. Recently the case of critical growth in the sense of Trudinger-Moser inequality in \mathbb{R}^2 was also considered in [3], there the authors studied the existence and concentration of the ground states.

The aim of the present paper is to consider the nonlinear Choquard equation with potential well, that is

$$(1.8) \quad \begin{cases} -\Delta u + (\lambda V(x) - \beta)u = (|x|^{-\mu} * |u|^{2_\mu^*})|u|^{2_\mu^*-2}u & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

where $\lambda, \beta \in \mathbb{R}^+$, $0 < \mu < N$, $N \geq 4$ and the potential V satisfies the assumptions:

(V₁) $V \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$, $V \geq 0$, and $\Omega := \text{int } V^{-1}(0)$ is a nonempty bounded set with smooth boundary, 0 is in interior of Ω and $\overline{\Omega} = V^{-1}(0)$.

(V₂) There exists $M_0 > 0$ such that

$$\mathcal{L}\{x \in \mathbb{R}^N : V(x) \leq M_0\} < \infty,$$

where \mathcal{L} denotes the Lebesgue measure in \mathbb{R}^N .

As we all know, the local nonlinear Schrödinger equation with deepening potential well has also been widely investigated. Consider

$$(1.9) \quad -\Delta u + (\lambda V(x) - \beta)u = |u|^{p-2}u, \quad \text{in } \mathbb{R}^N,$$

where the potential $V(x)$ satisfies (V₁) and (V₂). In [8], the authors studied the subcritical case and proved the existence of a least energy solution of (1.9) for large λ . They also showed that the sequence of least energy solutions converges strongly to a least energy solution for a problem in bounded domain. Furthermore, they also obtained the existence of at least $\text{cat}(\Omega)$ positive solutions for large λ , where $\Omega = \text{int}(V^{-1}(0))$ and $\text{cat}(\Omega)$ stands for the category of the domain Ω .

The critical case was considered in [13], there the authors proved the existence and multiplicity of positive solutions which localize near the potential well for β small and λ large. Later, they also proved the existence of solutions which change sign exactly once in [14]. We also refer to [7] where the authors proved the existence of k solutions that may change sign for any k and λ large enough. Suppose that the potential $V(x)$ satisfies (1.2), (1.3) and the nonlinearity is of subcritical growth, the authors in [17] overcame the loss of compactness and applied the deformation flow arguments to build the multi-bump shaped solutions. Recently the existence of multi-bump shaped solutions for (1.9) with critical growth was also studied in [23, 24, 39], the main results there generalize and complement the theorems in [17]. We would also like to mention some related nonlocal problems in [26] and the references therein, there the existence of solutions of the nonlocal Schrödinger-Poisson system was investigated under the effect of critical growth assumption or potential well type function $V(x)$. It is then quite natural to ask how the appearance of the potential well will affect the existence of solutions of the critical Choquard equation (1.8) and what is the asymptotic behavior of the solutions as the parameter λ goes to infinity, does the same results established for local Schrödinger equation still hold for the critical Choquard equation?

To study equation (1.8) by variational methods, we introduce the energy functional defined by

$$J_{\lambda, \beta}(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + (\lambda V(x) - \beta)|u|^2) dx - \frac{1}{2 \cdot 2_\mu^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2_\mu^*} |u(y)|^{2_\mu^*}}{|x - y|^\mu} dx dy.$$

The Hardy-Littlewood-Sobolev inequality implies that $J_{\lambda, \beta}$ is well defined on $H^1(\mathbb{R}^N)$ and belongs to \mathcal{C}^1 . Then we see that u is a weak solution of (1.8) if and only if u is a critical point of the functional $J_{\lambda, \beta}$. Furthermore, a function u_0 is called a ground state of (1.8) if u_0 is a critical point of (1.8) and $J_{\lambda, \beta}(u_0) \leq J_{\lambda, \beta}(u)$ holds

for any critical point u of (1.8), i.e.

$$J_{\lambda,\beta}(u_0) = c := \inf \left\{ J_{\lambda,\beta}(u) : u \in H^1(\mathbb{R}^N) \setminus \{0\} \text{ is a critical point of (1.8)} \right\}.$$

In the following we will denote the sequence of eigenvalues of the operator $-\Delta$ on Ω with homogeneous Dirichlet boundary data by

$$0 < \beta_1 < \beta_2 \leq \dots \leq \beta_j \leq \beta_{j+1} \leq \dots$$

and $\beta_j \rightarrow +\infty$ as $j \rightarrow +\infty$. Notice that whether the parameter β lies in $(0, \beta_1)$ or not affect the functional $J_{\lambda,\beta}$ greatly. If $0 < \beta < \beta_1$, the operator $-\Delta + \lambda V(x) - \beta$ is positively definite in $H^1(\mathbb{R}^N)$. However, if $\beta > \beta_1$, the operator $-\Delta + \lambda V(x) - \beta$ might be indefinite in $H^1(\mathbb{R}^N)$. Moreover, the appearance of convolution type nonlinearities brings us a lot of difficulties and the techniques in [8, 24, 39] can not be applied to the Choquard equation directly. Thus, to look for solutions for equation (1.8), we need to develop new techniques to overcome the difficulties.

The first result is to establish the existence of ground state solutions and the asymptotic behavior of the solutions for (1.8) with $\beta \in (0, \beta_1)$. The result reads as

Theorem 1.3. *Suppose that assumptions (V_1) and (V_2) hold, $0 < \mu < N$, $N \geq 4$. Then, for any $\beta \in (0, \beta_1)$ there exists $\lambda_\beta > 0$ such that, for each $\lambda \geq \lambda_\beta$, equation (1.8) has at least one ground state solution u , where β_1 is the first eigenvalue of $-\Delta$ on Ω with boundary condition $u = 0$. Furthermore, for any sequences $\lambda_n \rightarrow \infty$, then every sequence of solutions $\{u_n\}$ of (1.8) satisfying $J_{\lambda,\beta}(u_n) \rightarrow c < \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}$ as $n \rightarrow \infty$, converges to a solution of*

$$(1.10) \quad \begin{cases} -\Delta u = \beta u + (|x|^{-\mu} * |u|^{2_\mu^*}) |u|^{2_\mu^*-2} u & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases}$$

Ω is defined as in (V_1) .

Next we will use the Lusternik-Schnirelmann category (see e.g. [41]) to characterize the multiplicity result.

Theorem 1.4. *Assume (V_1) and (V_2) hold, $0 < \mu < N$ and $N \geq 4$. Then, there exist $0 < \beta^* < \beta_1$ and for each $0 < \beta \leq \beta^*$ two numbers $\lambda_\beta > 0$ and $0 < c_\beta < \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}$ such that, if $\lambda \geq \lambda_\beta$, then (1.8) has at least $\text{cat}(\Omega)$ solutions with energy $J_{\lambda,\beta} \leq c_\beta$, where $\text{cat}(\Omega)$ is the category of the domain Ω .*

Finally we are interested in the critical Choquard equation (1.8) with indefinite potential. In this case we assume that $\beta > \beta_1, \beta \neq \beta_j$ for any $j > 1$ and introduce assumption

$$(V_3) \liminf_{|x| \rightarrow \infty} V(x) > 0.$$

The result says that

Theorem 1.5. *Suppose that assumptions (V_1) and (V_3) hold, $0 < \mu < 4$, $N \geq 4$. Then, for any $\beta > \beta_1, \beta \neq \beta_j, j > 1$, there exists $\lambda_\beta > 0$ such that, for each $\lambda \geq \lambda_\beta$, equation (1.8) has at least one ground state solution u_λ . Furthermore, for any sequences $\lambda_n \rightarrow \infty$, the solution sequence $\{u_{\lambda_n}\}$ has a subsequence converging to a ground state solution u of (1.10).*

Remark 1.6. *Obviously assumption (V_3) is stronger than assumption (V_2) . To see this, we only need to take $M_0 = \frac{1}{2} \liminf_{|x| \rightarrow \infty} V(x)$.*

Throughout this paper we write $|\cdot|_q$ for the $L^q(\mathbb{R}^N)$ -norm, $q \in [1, \infty]$ and always assume that conditions (V_1) and (V_2) hold in Sections 2-4, conditions (V_1) and (V_3) hold in Sections 5-6, $0 < \mu < N$ and $N \geq 4$. We denote by C, C_1, C_2, C_3, \dots the different positive constants and

$$\|u\|_{H^1}^2 := \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) dx$$

the standard norm on $H^1(\mathbb{R}^N)$.

An outline of the paper is as follows: In Section 2, we give some preliminary results for the case $0 < \beta < \beta_1$ and prove Palais-Smale condition ((PS) condition, for short). In Section 3, we prove the existence of ground states for (1.8) by a problem on bounded region and show the certain concentration behavior of the solutions occurs as $\lambda \rightarrow \infty$. In Section 4, the Lusternik-Schnirelmann theory would give the existence of at least $cat(\Omega)$ critical points for (1.8). In Section 5, we give some preliminary results for the case $\beta > \beta_1$, $\beta \neq \beta_j$ for any $j > 1$. In Section 6, we prove the existence of ground states for (1.8) with indefinite potential and show the certain concentration behavior of the solutions occurs as $\lambda \rightarrow \infty$.

2. EXISTENCE OF SOLUTIONS FOR THE CASE $0 < \beta < \beta_1$

Next we denote by

$$E = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V u^2 dx < +\infty \right\}$$

the Hilbert space equipped with norm

$$\|u\| = \left(\|u\|_{H^1}^2 + \int_{\mathbb{R}^N} V u^2 dx \right)^{\frac{1}{2}}.$$

If $\lambda > 0$, then it is equivalent to the norms

$$\|u\|_\lambda = \left(\|u\|_{H^1}^2 + \lambda \int_{\mathbb{R}^N} V u^2 dx \right)^{\frac{1}{2}}.$$

Obviously, $H_0^1(\Omega) \subset E$, where Ω is defined as in (V_1) .

We denote the operator $L_{\lambda,\beta} := -\Delta + \lambda V(x) - \beta$ and particularly, $L_{\lambda,0} := -\Delta + \lambda V(x)$ and $L_{0,\beta} := -\Delta - \beta$. Observe that

$$0 \leq a_\lambda = \inf \{ \langle L_{\lambda,0} u, u \rangle : u \in E, |u|_2 = 1 \}$$

and that a_λ is nondecreasing in λ .

The following two Lemmas are taken from [13].

Lemma 2.1. *If $u_n \in E$ be such that $\lambda_n \rightarrow \infty$ and $\|u_n\|_{\lambda_n}^2 < C$. Then, there is a $u \in H_0^1(\Omega)$ such that, up to a subsequence, $u_n \rightarrow u$ in $L^2(\mathbb{R}^N)$.*

Lemma 2.2. *For every $0 < \beta < \beta_1$, there exists $\lambda_\beta > 0$ such that $a_\lambda \geq (\beta + \beta_1)/2$ for $\lambda \geq \lambda_\beta$. Consequently,*

$$C_\beta \|u\|_\lambda^2 \leq \langle L_{\lambda,\beta} u, u \rangle$$

for all $u \in E$, $\lambda \geq \lambda_\beta$, where $C_\beta > 0$ is a constant.

It follows from Lemma 2.2 that the operator $L_{\lambda,\beta}$ is positive if $\lambda \geq \lambda_\beta$ and thus we can introduce on E a new inner product

$$(u_1, u_2) = \langle L_{\lambda,\beta}^{\frac{1}{2}} u_1, L_{\lambda,\beta}^{\frac{1}{2}} u_2 \rangle$$

with the norm

$$\|u\|_{L_{\lambda,\beta}} = (u, u)^{\frac{1}{2}}.$$

Moreover, noting that for $\beta > 0$,

$$\|u\|_{L_{\lambda,\beta}} \leq \|u\|_\lambda, \quad \forall u \in E,$$

we know $\|u\|_{L_{\lambda,\beta}}$ in fact is equivalent to the norm $\|u\|_\lambda$ on E if $\lambda \geq \lambda_\beta$. For future use, enlarging λ_β if necessary, we may assume that $\lambda_\beta \geq \beta/M_0$, thus

$$(2.1) \quad \lambda M_0 - \beta \geq 0 \quad \text{for all } \lambda \geq \lambda_\beta,$$

where M_0 is given in (V_2) .

Since we are considering the critical case, we need to show where the compactness condition is recovered.

Proposition 2.3. For each $0 < \beta < \beta_1$ and $\lambda \geq \lambda_\beta$, $J_{\lambda,\beta}$ satisfies the $(PS)_c$ condition for all $c < \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}$.

Proof. Let $\{u_j\}$ be a $(PS)_c$ sequence, i.e.

$$(2.2) \quad J_{\lambda,\beta}(u_j) \rightarrow c$$

and

$$(2.3) \quad \sup\{|\langle J'_{\lambda,\beta}(u_j), \varphi \rangle| : \varphi \in E, \|\varphi\|_{L_{\lambda,\beta}} = 1\} \rightarrow 0$$

as $j \rightarrow +\infty$. By (2.2) and (2.3), for any $j \in \mathbb{N}$, it easily follows that there exists $C_1 > 0$ such that

$$(2.4) \quad |J_{\lambda,\beta}(u_j)| \leq C_1$$

and

$$(2.5) \quad |\langle J'_{\lambda,\beta}(u_j), \frac{u_j}{\|u_j\|_{L_{\lambda,\beta}}} \rangle| \leq C_1.$$

Consequently, we have

$$(2.6) \quad \begin{aligned} \frac{N+2-\mu}{4N-2\mu} \langle L_{\lambda,\beta} u_j, u_j \rangle &= J_{\lambda,\beta}(u_j) - \frac{1}{2 \cdot 2_\mu^*} \langle J'_{\lambda,\beta}(u_j), u_j \rangle \\ &\leq C_1(1 + \|u_j\|_{L_{\lambda,\beta}}), \end{aligned}$$

that is

$$\|u_j\|_{L_{\lambda,\beta}}^2 \leq C_2(1 + \|u_j\|_{L_{\lambda,\beta}}),$$

which means $\{u_j\}$ is bounded in E .

Now, up to a subsequence, still denoted by $\{u_j\}$, we may assume that there exists $u_\infty \in E$ such that $u_j \rightharpoonup u_\infty$ in E and

$$(2.7) \quad u_j \rightarrow u_\infty \quad a.e. \text{ in } \mathbb{R}^N$$

as $j \rightarrow +\infty$. From the fact that $|u_j|^{2_\mu^*}$ is bounded in $L^{\frac{2N}{2N-2\mu}}(\mathbb{R}^N)$ we have

$$|u_j|^{2_\mu^*} \rightharpoonup |u_\infty|^{2_\mu^*} \quad \text{in } L^{\frac{2N}{2N-2\mu}}(\mathbb{R}^N)$$

as $j \rightarrow +\infty$. By the Hardy-Littlewood-Sobolev inequality, the Riesz potential defines a linear continuous map from $L^{\frac{2N}{2N-2\mu}}(\mathbb{R}^N)$ to $L^{\frac{2N}{2\mu}}(\mathbb{R}^N)$, we know that

$$\int_{\mathbb{R}^N} \frac{|u_j(y)|^{2_\mu^*}}{|x-y|^\mu} dy \rightharpoonup \int_{\mathbb{R}^N} \frac{|u_\infty(y)|^{2_\mu^*}}{|x-y|^\mu} dy \quad \text{in } L^{\frac{2N}{2\mu}}(\mathbb{R}^N)$$

as $j \rightarrow +\infty$. Combining this with the fact that

$$|u_j|^{2_\mu^*-2} u_j \rightharpoonup |u_\infty|^{2_\mu^*-2} u_\infty \quad \text{in } L^{\frac{2N}{N-\mu+2}}(\mathbb{R}^N)$$

as $j \rightarrow +\infty$, we have

$$(2.8) \quad \int_{\mathbb{R}^N} \frac{|u_j(y)|^{2_\mu^*}}{|x-y|^\mu} dy |u_j(x)|^{2_\mu^*-2} u_j(x) \rightharpoonup \int_{\mathbb{R}^N} \frac{|u_\infty(y)|^{2_\mu^*}}{|x-y|^\mu} dy |u_\infty(x)|^{2_\mu^*-2} u_\infty(x) \quad \text{in } L^{\frac{2N}{N+2}}(\mathbb{R}^N)$$

as $j \rightarrow +\infty$. Since, for any $\varphi \in E$ $\langle J'_{\lambda,\beta}(u_j), \varphi \rangle \rightarrow 0$, passing to the limit as $j \rightarrow +\infty$ and taking into account (2.8) we get

$$\int_{\mathbb{R}^N} (\nabla u_\infty \nabla \varphi + (\lambda V(x) - \beta) u_\infty \varphi) dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_\infty(x)|^{2_\mu^*} |u_\infty(y)|^{2_\mu^*-2} u_\infty(y) \varphi(y)}{|x-y|^\mu} dx dy$$

for any $\varphi \in E$, that means u_∞ is a solution of problem (1.8). Moreover, taking $\varphi = u_\infty \in E$ as a test function in (1.8), we have

$$\int_{\mathbb{R}^N} (|\nabla u_\infty|^2 + (\lambda V(x) - \beta) u_\infty^2) dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_\infty(x)|^{2_\mu^*} |u_\infty(y)|^{2_\mu^*}}{|x-y|^\mu} dx dy,$$

thus

$$J_{\lambda,\beta}(u_\infty) = \frac{N+2-\mu}{4N-2\mu} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_\infty(x)|^{2_\mu^*} |u_\infty(y)|^{2_\mu^*}}{|x-y|^\mu} dx dy \geq 0.$$

Now, we write $v_j := u_j - u_\infty$, then, $v_j \rightharpoonup 0$ in E and $v_j \rightarrow 0$ a.e. in \mathbb{R}^N . By the Brézis-Lieb type splitting result for nonlocal term in [18] which says

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_j(x)|^{2_\mu^*} |u_j(y)|^{2_\mu^*}}{|x-y|^\mu} dx dy = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_j(x)|^{2_\mu^*} |v_j(y)|^{2_\mu^*}}{|x-y|^\mu} dx dy + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_\infty(x)|^{2_\mu^*} |u_\infty(y)|^{2_\mu^*}}{|x-y|^\mu} dx dy + o_j(1)$$

as $j \rightarrow +\infty$, we know that

$$\begin{aligned} c &\leftarrow J_{\lambda,\beta}(u_j) \\ &= \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla v_j|^2 + (\lambda V(x) - \beta) v_j^2) dx + \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u_\infty|^2 + (\lambda V(x) - \beta) u_\infty^2) dx \\ &\quad - \frac{1}{2 \cdot 2_\mu^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_j(x)|^{2_\mu^*} |v_j(y)|^{2_\mu^*}}{|x-y|^\mu} dx dy - \frac{1}{2 \cdot 2_\mu^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_\infty(x)|^{2_\mu^*} |u_\infty(y)|^{2_\mu^*}}{|x-y|^\mu} dx dy + o_j(1) \\ &= J_{\lambda,\beta}(u_\infty) + J_{\lambda,\beta}(v_j) + o_j(1). \end{aligned} \tag{2.9}$$

Analogously, we have

$$\langle J'_{\lambda,\beta}(u_j), u_j \rangle = \langle J'_{\lambda,\beta}(u_\infty), u_\infty \rangle + \langle J'_{\lambda,\beta}(v_j), v_j \rangle + o_j(1).$$

It follows from $\langle J'_{\lambda,\beta}(u_\infty), u_\infty \rangle = 0$ and $\langle J'_{\lambda,\beta}(u_j), u_j \rangle \rightarrow 0$ that

$$\int_{\mathbb{R}^N} (|\nabla v_j|^2 + (\lambda V(x) - \beta) v_j^2) dx \rightarrow b \quad \text{and} \quad \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_j(x)|^{2_\mu^*} |v_j(y)|^{2_\mu^*}}{|x-y|^\mu} dx dy \rightarrow b.$$

Since $J_{\lambda,\beta}(u_\infty) \geq 0$ and (2.9), we obtain,

$$c \geq \frac{N+2-\mu}{4N-2\mu} b. \tag{2.10}$$

By Lemma 2.1 one knows that as $j \rightarrow \infty$, $\int_F |v_j|^2 dx \rightarrow 0$, where $F = \{x \in \mathbb{R}^N : V(x) \leq M_0\}$. Let $F^c = \mathbb{R}^N \setminus F$. Then, from the definition of $S_{H,L}$ and (2.1), we have

$$\begin{aligned} S_{H,L} \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_j(x)|^{2_\mu^*} |v_j(y)|^{2_\mu^*}}{|x-y|^\mu} dx dy \right)^{\frac{N-2}{2N-\mu}} &\leq \int_{\mathbb{R}^N} |\nabla v_j|^2 dx \\ &\leq \int_{\mathbb{R}^N} |\nabla v_j|^2 dx + \int_{F^c} (\lambda V(x) - \beta) |v_j|^2 dx \\ &\leq \int_{\mathbb{R}^N} (|\nabla v_j|^2 + (\lambda V(x) - \beta) |v_j|^2) dx + \beta \int_F |v_j|^2 dx \\ &= \int_{\mathbb{R}^N} (|\nabla v_j|^2 + (\lambda V(x) - \beta) |v_j|^2) dx + o_j(1), \end{aligned}$$

passing to the limit, it yields that $b \geq S_{H,L} b^{\frac{N-2}{2N-\mu}}$. Then we have either $b = 0$ or $b \geq S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}$. If $b = 0$, the proof is complete. Otherwise $b \geq S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}$, then we can obtain from (2.10),

$$\frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} \leq \frac{N+2-\mu}{4N-2\mu} b \leq c,$$

which contradicts with the fact that $c < \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}$. Thus $b = 0$, and

$$\|u_j - u_\infty\|_{L_{\lambda,\beta}} \rightarrow 0$$

as $j \rightarrow +\infty$. This ends the proof of Proposition 2.3. \square

3. PROOF OF THEOREM 1.3

It is convenient to show that the functional $J_{\lambda,\beta}$ satisfies the Mountain-Pass geometry.

Lemma 3.1. *For any $0 < \beta < \beta_1$, $\lambda > 0$ large enough, the functional $J_{\lambda,\beta}$ satisfies the following conditions.*

- (i) *There exist $\alpha, \rho > 0$ such that $J_{\lambda,\beta}(u) \geq \alpha$ for $\|u\|_{L_{\lambda,\beta}} = \rho$.*
- (ii) *There exists a $w_1 \in E$ with $\|w_1\|_{L_{\lambda,\beta}} > \rho$ such that $J_{\lambda,\beta}(w_1) < 0$.*

Proof. (i) By $0 < \beta < \beta_1$, the Sobolev embedding and Hardy-Littlewood-Sobolev inequality, for all $u \in E \setminus \{0\}$ we have

$$\begin{aligned} J_{\lambda,\beta}(u) &\geq \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + (\lambda V(x) - \beta)|u|^2) dx - \frac{1}{2 \cdot 2_\mu^*} C_1 |u|_{2_\mu^*}^{2 \cdot 2_\mu^*} \\ &\geq C_2 \|u\|_{L_{\lambda,\beta}}^2 - C_3 \|u\|_{L_{\lambda,\beta}}^{2 \cdot 2_\mu^*}. \end{aligned}$$

Since $2 < 2 \cdot 2_\mu^*$, we can choose some $\alpha, \rho > 0$ such that $J_{\lambda,\beta}(u) \geq \alpha$ for $\|u\|_{L_{\lambda,\beta}} = \rho$.

(ii) For any $u_1 \in E \setminus \{0\}$, we have

$$J_{\lambda,\beta}(tu_1) = \frac{t^2}{2} \int_{\mathbb{R}^N} (|\nabla u_1|^2 + (\lambda V(x) - \beta)u_1^2) dx - \frac{t^{2 \cdot 2_\mu^*}}{2 \cdot 2_\mu^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_1(x)|^{2_\mu^*} |u_1(y)|^{2_\mu^*}}{|x-y|^\mu} dx dy < 0$$

for $t > 0$ large enough. Hence, we can take a $w_1 := t_1 u_1$ for some $t_1 > 0$ and (ii) follows. \square

Applying the mountain pass theorem without (PS) condition (cf. [41]), there exists a (PS) sequence $\{u_n\}$ such that $J_{\lambda,\beta}(u_n) \rightarrow c$ and $J'_{\lambda,\beta}(u_n) \rightarrow 0$ in E^{-1} at the minimax level

$$c_{\lambda,\beta} = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_{\lambda,\beta}(\gamma(t)) > 0,$$

where

$$\Gamma := \{\gamma \in C([0,1], E) : \gamma(0) = 0, J_{\lambda,\beta}(\gamma(1)) < 0\}.$$

If we denote the Nehari manifold of $J_{\lambda,\beta}$ by

$$\mathcal{M}_{\lambda,\beta} = \{u \in E \setminus \{0\} : \langle J'_{\lambda,\beta}(u), u \rangle = 0\},$$

since $0 < \beta < \beta_1$ and $2 < 2 \cdot 2_\mu^*$, the function $t \in \mathbb{R}_+ \rightarrow J_{\lambda,\beta}(tu)$ has an unique maximum point $t(u) > 0$ and $t(u)u \in \mathcal{M}_{\lambda,\beta}$. Then $c_{\lambda,\beta}$ has an equivalent minimax characterization, that is

$$(3.1) \quad c_{\lambda,\beta} := \inf_{u \in \mathcal{M}_{\lambda,\beta}} J_{\lambda,\beta}(u) = \inf_{u \in E, u \neq 0} \max_{t \geq 0} J_{\lambda,\beta}(tu).$$

Next we denote by $J_{\beta,\Omega}$ the restriction of $J_{\lambda,\beta}$ on $H_0^1(\Omega)$, that is

$$J_{\beta,\Omega}(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{\beta}{2} \int_{\Omega} |u|^2 dx - \frac{1}{2 \cdot 2_\mu^*} \int_{\Omega} \int_{\Omega} \frac{|u(x)|^{2_\mu^*} |u(y)|^{2_\mu^*}}{|x-y|^\mu} dx dy,$$

where Ω is defined as in (V_1) . The Nehari manifold of $J_{\beta,\Omega}$ is

$$\mathcal{M}_{\beta,\Omega} = \{u \in H_0^1(\Omega) \setminus \{0\} : \langle J'_{\beta,\Omega}(u), u \rangle = 0\}.$$

Set

$$c(\beta, \Omega) := \inf_{u \in \mathcal{M}_{\beta,\Omega}} J_{\beta,\Omega}(u).$$

Analogously, we have

$$(3.2) \quad c(\beta, \Omega) = \inf_{u \in H_0^1(\Omega), u \neq 0} \max_{t \geq 0} J_{\beta,\Omega}(tu) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_{\beta,\Omega}(\gamma(t)),$$

where

$$\Gamma := \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0, J_{\beta,\Omega}(\gamma(1)) < 0\}.$$

The following Lemma will plays an important role in estimating the Mountain pass levels. By the proof of Theorem 1.4 (i) in [18], we have

Lemma 3.2. *Let $\beta > 0$, $\beta \neq \beta_j$ for any $j \geq 1$. There exists $e \in H_0^1(\Omega) \setminus \{0\}$ such that*

$$(3.3) \quad \sup_{t \geq 0} J_{\beta, \Omega}(te) < \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}.$$

Proposition 3.3. *Let $\beta > 0$, $\beta \neq \beta_j$ for any $j \geq 1$. If $\lambda \geq \lambda_\beta$ then*

$$(3.4) \quad 0 < c_{\lambda, \beta} \leq c(\beta, \Omega) < \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}.$$

Proof. Lemma 3.1 implies $c_{\lambda, \beta} > 0$. Since

$$\left\{ u \in H_0^1(\Omega) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2_\mu^*} |u(y)|^{2_\mu^*}}{|x-y|^\mu} dx dy = 1 \right\} \subset \left\{ u \in E : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2_\mu^*} |u(y)|^{2_\mu^*}}{|x-y|^\mu} dx dy = 1 \right\}$$

and $\langle L_{\lambda, \beta} u, u \rangle = \langle L_{0, \beta} u, u \rangle$ for

$$u \in \left\{ u \in H_0^1(\Omega) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2_\mu^*} |u(y)|^{2_\mu^*}}{|x-y|^\mu} dx dy = 1 \right\},$$

it follows that $c_{\lambda, \beta} \leq c(\beta, \Omega)$. By Lemma 3.2 and (3.2), we know $c(\beta, \Omega) < \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}$. Hence, the conclusion is proved. \square

Proof of Theorem 1.3. Applying the Mountain-Pass theorem without (PS) condition, we know there exists a $(PS)_{c_{\lambda, \beta}}$ sequence $\{u_n\}$. Then we obtain from Proposition 2.3 and Proposition 3.3, (1.8) has at least one ground state solution u .

In the following, we come to give the asymptotic behavior of the solutions of (1.8) as λ goes to infinity. For $0 < \beta < \beta_1$, let $\{u_n\}$ be a sequence of solutions of (1.8) such that $\lambda_n \rightarrow \infty$ and $J_{\lambda_n, \beta}(u_n) \rightarrow c < \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}$, we have

$$J_{\lambda_n, \beta}(u_n) = \frac{N+2-\mu}{4N-2\mu} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2_\mu^*} |u_n(y)|^{2_\mu^*}}{|x-y|^\mu} dx dy,$$

and so,

$$(3.5) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2_\mu^*} |u_n(y)|^{2_\mu^*}}{|x-y|^\mu} dx dy < S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}.$$

By Lemma 2.2, we can deduce that

$$\frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}} > J_{\lambda_n, \beta}(u_n) = \frac{N+2-\mu}{4N-2\mu} \langle L_{\lambda_n, \beta} u_n, u_n \rangle \geq \frac{N+2-\mu}{4N-2\mu} C_\beta \|u_n\|_{L_{\lambda_n, \beta}}^2,$$

and so $\{u_n\}$ is bounded in E .

By Lemma 2.1, there is a $u \in H_0^1(\Omega)$ such that, up to a subsequence, $u_n \rightharpoonup u$ in E and $u_n \rightarrow u$ in $L^2(\mathbb{R}^N)$. From the fact that u_n is a solution of (1.8), we have

$$\int_{\mathbb{R}^N} (\nabla u_n \nabla \varphi + (\lambda_n V - \beta) u_n \varphi) dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2_\mu^*} |u_n(y)|^{2_\mu^* - 2} u_n(y) \varphi(y)}{|x-y|^\mu} dx dy$$

for any $\varphi \in E$. If $\varphi \in H_0^1(\Omega)$ then $\lambda_n \int_{\mathbb{R}^N} V u_n \varphi dx = 0$ for all n . Letting $n \rightarrow \infty$ we obtain

$$\int_{\mathbb{R}^N} (\nabla u \nabla \varphi - \beta u \varphi) dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2_\mu^*} |u(y)|^{2_\mu^* - 2} u(y) \varphi(y)}{|x-y|^\mu} dx dy$$

for any $\varphi \in H_0^1(\Omega)$. So, u is a solution of (1.10). Define $v_n := u_n - u$, then $v_n \rightarrow 0$ in $L^2(\mathbb{R}^N)$ and $v_n \rightarrow 0$ a.e. in \mathbb{R}^N as $n \rightarrow +\infty$.

Since $V(x) = 0$ for $x \in \Omega$, we get

$$(3.6) \quad \langle L_{\lambda_n, \beta} u_n, u_n \rangle = \langle L_{0, \beta} u, u \rangle + \langle L_{\lambda_n, \beta} v_n, v_n \rangle.$$

Since $\{u_n\}$ is a sequence of solutions of (1.8) and u is a solution of (1.10), by the Brézis-Lieb type splitting result for nonlocal term in [18] that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2^*_\mu} |u_n(y)|^{2^*_\mu}}{|x-y|^\mu} dx dy = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\mu} |u(y)|^{2^*_\mu}}{|x-y|^\mu} dx dy + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n(x)|^{2^*_\mu} |v_n(y)|^{2^*_\mu}}{|x-y|^\mu} dx dy + o_n(1),$$

we can get

$$(3.7) \quad \langle L_{\lambda_n, \beta} v_n, v_n \rangle - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n(x)|^{2^*_\mu} |v_n(y)|^{2^*_\mu}}{|x-y|^\mu} dx dy = o_n(1).$$

We claim that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n(x)|^{2^*_\mu} |v_n(y)|^{2^*_\mu}}{|x-y|^\mu} dx dy \rightarrow 0.$$

Assume by contrary that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n(x)|^{2^*_\mu} |v_n(y)|^{2^*_\mu}}{|x-y|^\mu} dx dy \rightarrow b > 0.$$

Then,

$$\begin{aligned} S_{H,L} & \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n(x)|^{2^*_\mu} |v_n(y)|^{2^*_\mu}}{|x-y|^\mu} dx dy \right)^{\frac{N-2}{2N-\mu}} \\ & \leq \int_{\mathbb{R}^N} |\nabla v_n|^2 dx \\ & \leq \int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \int_{F^c} \lambda_n V(x) |v_n|^2 dx - \beta \int_{F^c} |v_n|^2 dx \\ & = \int_{\mathbb{R}^N} (|\nabla v_n|^2 dx + \lambda_n V |v_n|^2 - \beta |v_n|^2) dx + \beta \int_F |v_n|^2 dx \\ & = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n(x)|^{2^*_\mu} |v_n(y)|^{2^*_\mu}}{|x-y|^\mu} dx dy + o_n(1), \end{aligned}$$

thanks to (2.1) and (3.7). It follows that

$$S_{H,L} \leq \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n(x)|^{2^*_\mu} |v_n(y)|^{2^*_\mu}}{|x-y|^\mu} dx dy \right)^{\frac{N-\mu+2}{2N-\mu}} + o_n(1) \leq \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2^*_\mu} |u_n(y)|^{2^*_\mu}}{|x-y|^\mu} dx dy \right)^{\frac{N-\mu+2}{2N-\mu}} + o_n(1)$$

and so, by (3.5),

$$S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} \leq \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2^*_\mu} |u_n(y)|^{2^*_\mu}}{|x-y|^\mu} dx dy < S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}.$$

This is a contradiction and consequently

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n(x)|^{2^*_\mu} |v_n(y)|^{2^*_\mu}}{|x-y|^\mu} dx dy \rightarrow 0.$$

From (3.7) we get

$$(3.8) \quad \langle L_{\lambda_n, \beta} v_n, v_n \rangle \rightarrow 0.$$

Hence, by (3.6)

$$(3.9) \quad \lim_{n \rightarrow \infty} \langle L_{\lambda_n, \beta} u_n, u_n \rangle = \langle L_{0, \beta} u, u \rangle.$$

Recall that $\int_{\mathbb{R}^N} |\nabla v_n|^2 dx \geq \beta \int_{\mathbb{R}^N} |v_n|^2 dx$, we know

$$\begin{aligned} \int_{\mathbb{R}^N} V |u_n|^2 dx & \leq \int_{\mathbb{R}^N} \lambda_n V |u_n|^2 dx \\ & = \int_{\mathbb{R}^N} \lambda_n V |v_n|^2 dx \\ & \leq \langle L_{\lambda_n, \beta} v_n, v_n \rangle, \end{aligned}$$

since $u_n = v_n$ in $\mathbb{R}^N \setminus \Omega$ and $V = 0$ for $x \in \Omega$. Combining this with (3.8), we know $\int_{\mathbb{R}^N} V|u_n|^2 dx \rightarrow 0$ and obtain from (3.9) that $u_n \rightarrow u$ in E . \square

Remark 3.4. From Theorem 1.3, we know that for every $0 < \beta < \beta_1$ there exists $\lambda_\beta > 0$ such that, for each $\lambda \geq \lambda_\beta$, equation (1.8) has at least one ground state solution u . Let $\lambda_n \geq \lambda_\beta$ and $\lambda_n \rightarrow \infty$, we denote $\{u_n\}$ be a sequence of ground state solutions of (1.8) with $\lambda = \lambda_n$. By Proposition 3.3, we have $J_{\lambda_n, \beta}(u_n) \leq c(\beta, \Omega) < \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}$. It is easy to see that $\{u_n\}$ is bounded in E , $\lambda_n \rightarrow \infty$ and $J_{\lambda_n, \beta}(u_n) \rightarrow c < \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}$.

Remark 3.5. From the main results in [18], we can see that $c(\beta, \Omega)$ can also be achieved by a function, hereafter it will be denoted by u_β on the Nehari manifold $\mathcal{M}_{\beta, \Omega}$. Moreover, we add a subscript r to denote the same quantities when the domain Ω is replaced by $B_r \subset \Omega$. Then, $c(\beta, B_r)$ can also be achieved by some $u_{\beta, B_r} \in \mathcal{M}_{\beta, B_r}$.

4. MULTIPLICITY OF SOLUTIONS FOR THE CASE $0 < \beta < \beta_1$

We consider

$$(4.1) \quad \begin{cases} -\Delta u = (|x|^{-\mu} * |u|^{2_\mu^*}) |u|^{2_\mu^*-2} u & \text{in } \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N). \end{cases}$$

The functional associated to (4.1) is

$$J_*(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{2 \cdot 2_\mu^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2_\mu^*} |u(y)|^{2_\mu^*}}{|x-y|^\mu} dx dy$$

and its Nehari manifold is

$$\mathcal{M}_* = \{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\} : \langle J'_*(u), u \rangle = 0\}.$$

Set

$$c_* := \inf_{u \in \mathcal{M}_*} J_*(u).$$

We can get

$$(4.2) \quad c_* = \inf_{u \in D^{1,2}(\mathbb{R}^N), u \neq 0} \max_{t \geq 0} J_*(tu),$$

moreover, since $\tilde{U}(x)$ is the unique solution, we know

$$(4.3) \quad c_* = \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}.$$

Lemma 4.1. Let $0 < \beta < \beta_1$ and t_β be the unique value such that $t_\beta u_\beta \in \mathcal{M}_*$. Then

$$\lim_{\beta \rightarrow 0} t_\beta = 1.$$

Where u_β is defined in Remark 3.5.

Proof. By the definition of \mathcal{M}_* , t_β satisfies

$$t_\beta^2 \int_{\mathbb{R}^N} |\nabla u_\beta|^2 dx = t_\beta^{2 \cdot 2_\mu^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_\beta(x)|^{2_\mu^*} |u_\beta(y)|^{2_\mu^*}}{|x-y|^\mu} dx dy.$$

By Remark 3.5, we have

$$\int_{\mathbb{R}^N} |\nabla u_\beta|^2 dx = \beta \int_{\mathbb{R}^N} |u_\beta|^2 dx + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_\beta(x)|^{2_\mu^*} |u_\beta(y)|^{2_\mu^*}}{|x-y|^\mu} dx dy.$$

From the two equalities above, we get

$$\limsup_{\beta \rightarrow 0} t_\beta \geq 1.$$

By Proposition 3.3, we have

$$\begin{aligned}
& \left(\frac{1}{2} - \frac{1}{2 \cdot 2_\mu^*} \right) \left[\int_{\mathbb{R}^N} |\nabla u_\beta|^2 dx - \beta \int_{\mathbb{R}^N} |u_\beta|^2 dx \right] \\
&= J_{\beta, \Omega}(u_\beta) - \frac{1}{2 \cdot 2_\mu^*} \langle J'_{\beta, \Omega}(u_\beta), u_\beta \rangle \\
&\leq \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}.
\end{aligned}$$

Since $0 < \beta < \beta_1$, $\int_{\mathbb{R}^N} |\nabla u_\beta|^2 dx$ is bounded uniformly in β . Then,

$$\int_{\mathbb{R}^N} |\nabla u_\beta|^2 dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_\beta(x)|^{2_\mu^*} |u_\beta(y)|^{2_\mu^*}}{|x-y|^\mu} dx dy + o_\beta(1)$$

as $\beta > 0$ small enough. Thus, $\lim_{\beta \rightarrow 0} t_\beta = 1$. \square

Without loss of generality, we may assume that $B_\delta \subset \Omega \subset B_{\kappa_0 \delta}$ for some positive κ_0 . Consider a cut-off function $\psi \in C_0^\infty(\mathbb{R}^N)$ such that

$$\psi(x) = 1 \quad \text{if } |x| \leq \delta, \quad \psi(x) = 0 \quad \text{if } |x| \geq 2\delta.$$

We define, for $\varepsilon > 0$,

$$\begin{aligned}
(4.4) \quad & U_\varepsilon(x) := \varepsilon^{\frac{2-N}{2}} U\left(\frac{x}{\varepsilon}\right), \\
& u_\varepsilon(x) := \psi(x) U_\varepsilon(x),
\end{aligned}$$

where U is defined in introduction. From [18, 20] and Lemma 1.46 of [41], we know that as $\varepsilon \rightarrow 0^+$,

$$(4.5) \quad \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 dx = C(N, \mu)^{\frac{N-2}{2N-\mu} \cdot \frac{N}{2}} S_{H,L}^{\frac{N}{2}} + O(\varepsilon^{N-2}),$$

$$(4.6) \quad \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_\varepsilon(x)|^{2_\mu^*} |u_\varepsilon(y)|^{2_\mu^*}}{|x-y|^\mu} dx dy \right)^{\frac{N-2}{2N-\mu}} \leq C(N, \mu)^{\frac{N-2}{2N-\mu} \cdot \frac{N}{2}} S_{H,L}^{\frac{N-2}{2}} + O(\varepsilon^{N-2})$$

and

$$(4.7) \quad \int_{\mathbb{R}^N} |u_\varepsilon|^2 dx = \begin{cases} d\varepsilon^2 |\ln \varepsilon| + O(\varepsilon^2) & \text{if } N = 4, \\ d\varepsilon^2 + O(\varepsilon^{N-2}) & \text{if } N \geq 5, \end{cases}$$

where d is a positive constant.

Lemma 4.2. $\lim_{\beta \rightarrow 0} c_{\beta, \Omega} = c_*$. Where $c_{\beta, \Omega}$ is defined as in (3.2).

Proof. If $N \geq 5$, by (4.4) to (4.7), for ε small enough, we have

$$\begin{aligned}
\max_{t \geq 0} J_{\beta, \Omega}(tu_\varepsilon) &= \max_{t \geq 0} \left(\frac{t^2}{2} \int_{\Omega} |\nabla u_\varepsilon|^2 dx - \frac{\beta t^2}{2} \int_{\Omega} |u_\varepsilon|^2 dx - \frac{t^{2 \cdot 2_\mu^*}}{2 \cdot 2_\mu^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_\varepsilon(x)|^{2_\mu^*} |u_\varepsilon(y)|^{2_\mu^*}}{|x-y|^\mu} dx dy \right) \\
&= \frac{N+2-\mu}{4N-2\mu} \left[\frac{\int_{\Omega} |\nabla u_\varepsilon|^2 dx - \beta \int_{\Omega} |u_\varepsilon|^2 dx}{\left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_\varepsilon(x)|^{2_\mu^*} |u_\varepsilon(y)|^{2_\mu^*}}{|x-y|^\mu} dx dy \right)^{\frac{N-2}{2N-\mu}}} \right]^{\frac{2N-\mu}{N+2-\mu}} \\
&\geq \frac{N+2-\mu}{4N-2\mu} \left[\frac{C(N, \mu)^{\frac{N-2}{2N-\mu} \cdot \frac{N}{2}} S_{H,L}^{\frac{N}{2}} + O(\varepsilon^{N-2}) - \beta O(\varepsilon^2)}{C(N, \mu)^{\frac{N-2}{2N-\mu} \cdot \frac{N}{2}} S_{H,L}^{\frac{N-2}{2}} + O(\varepsilon^{N-2})} \right]^{\frac{2N-\mu}{N+2-\mu}} \\
&= \frac{N+2-\mu}{4N-2\mu} [S_{H,L} - \beta O(\varepsilon^2)]^{\frac{2N-\mu}{N+2-\mu}}.
\end{aligned}$$

Then, we have

$$\lim_{\beta \rightarrow 0} \max_{t \geq 0} J_{\beta, \Omega}(tu_\varepsilon) \geq \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}$$

for ε small enough. Similarly, if $N = 4$, we have

$$\lim_{\beta \rightarrow 0} \max_{t \geq 0} J_{\beta, \Omega}(tu_\varepsilon) \geq \frac{6-\mu}{16-2\mu} S_{H,L}^{\frac{8-\mu}{6-\mu}}$$

for ε small enough. So, by (3.2), we get

$$\lim_{\beta \rightarrow 0} c_{\beta, \Omega} \geq \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}},$$

that is,

$$\lim_{\beta \rightarrow 0} c_{\beta, \Omega} \geq c_*.$$

On the other hand, by Proposition 3.3 and (4.3), we already have

$$c_{\beta, \Omega} < c_*$$

for every $0 < \beta < \beta_1$. Hence the conclusion follows. \square

To prove Theorem 1.4, we follow the idea in [9]. The barycenter of function $u \in H_0^1(\Omega)$ (see [9]) is defined as

$$\alpha(u) = \frac{\int_{\mathbb{R}^N} x |\nabla u|^2 dx}{\int_{\mathbb{R}^N} |\nabla u|^2 dx}.$$

Since Ω is a bounded smooth domain of \mathbb{R}^N , we may fix $r > 0$ small enough such that

$$\Omega_{2r}^+ = \{x \in \mathbb{R}^N : d(x, \Omega) \leq 2r\}$$

and

$$\Omega_r^- = \{x \in \Omega : d(x, \partial\Omega) \geq r\}$$

are homotopically equivalent to Ω . In particular we denote by

$$h : \Omega_{2r}^+ \rightarrow \Omega_r^-$$

the homotopic equivalence map such that $h|_{\Omega_r^-}$ is the identity.

Lemma 4.3. *Let $\{u_n\} \subset H_0^1(\Omega)$ be a (PS) sequence for J_* at level $c_* = \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}$. Then, for some subsequence of $\{u_n\}$, still denoted by itself, such that*

(i) $\{u_n\}$ has a subsequence strongly convergent in $D^{1,2}(\mathbb{R}^N)$; or

(ii) there exists $\{y_n\} \subset \Omega$ such that the sequence $v_n(x) = u_n(x + y_n)$ converges strongly in $D^{1,2}(\mathbb{R}^N)$.

Proof. By (2.6) with $\lambda = 0$ and $\beta = 0$, we know the sequence $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Hence, there exists $u \in H_0^1(\Omega)$ such that $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, up to some subsequence. We next continue our arguments by distinguishing two cases: $u \neq 0$ and $u = 0$.

Case 1. $u \neq 0$.

In this case, since $\{u_n\} \subset H_0^1(\Omega)$ is bounded and $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, we have

$$(4.8) \quad \int_{\mathbb{R}^N} |\nabla u|^2 dx \leq \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx.$$

Since $\{u_n\}$ is a (PS) sequence for J_* , we can get that $\langle J'_*(u), u \rangle = 0$. Observe that we must have the equality in (4.8). Otherwise, by Fatou's lemma,

$$\begin{aligned} c_* &\leq J_*(u) \\ &= J_*(u) - \frac{1}{2 \cdot 2_\mu^*} \langle J'_*(u), u \rangle \\ &= \frac{N+2-\mu}{4N-2\mu} \int_{\mathbb{R}^N} |\nabla u|^2 dx \\ &\leq \lim_{n \rightarrow \infty} \frac{N+2-\mu}{4N-2\mu} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \\ &= \lim_{n \rightarrow \infty} (J_*(u_n) - \frac{1}{2 \cdot 2_\mu^*} \langle J'_*(u_n), u_n \rangle) \leq c_*, \end{aligned}$$

which leads to a contradiction. Thus, up to subsequences, we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \rightarrow \int_{\mathbb{R}^N} |\nabla u|^2 dx.$$

Hence, $\{u_n\}$ has a subsequence which convergent to u strongly in $D^{1,2}(\mathbb{R}^N)$.

Case 2. $u = 0$.

Since $\{u_n\}$ is a (PS) sequence for J_* , we get

$$\begin{aligned} J_*(u_n) &= J_*(u_n) - \frac{1}{2 \cdot 2_\mu^*} \langle J'_*(u_n), u_n \rangle + o_n(1) \\ &= \frac{N+2-\mu}{4N-2\mu} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + o_n(1) \rightarrow \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}. \end{aligned}$$

Then $\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \not\rightarrow 0$. So, there exist $r, \delta > 0$ such that

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_r(y)} |\nabla u_n|^2 dx \geq \delta.$$

Otherwise, we have $\nabla u_n \rightarrow 0$ in $L^p(\mathbb{R}^N)$ with $2 < p < 2^*$ from the concentration compactness principle (see Lemma 1.21 of [41]). Since $\{u_n\} \subset H_0^1(\Omega)$ and Ω is bounded, we can deduce $\nabla u_n \rightarrow 0$ in $L^2(\mathbb{R}^N)$, which contradicts to the fact that $\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \not\rightarrow 0$. So, there exist $r, \delta > 0$ and $\{y_n\} \subset \mathbb{R}^N$ such that

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_r(y)} |\nabla u_n|^2 dx \geq \delta.$$

Since $\text{supp } u_n \subset \Omega$, we can choose $\{y_n\} \subset \Omega$. Let $v_n(x) = u_n(x + y_n)$, then $J_*(v_n) \rightarrow \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}$ and $\langle J'_*(v_n), v_n \rangle \rightarrow 0$. It is clear that v_n is bounded in $D^{1,2}(\mathbb{R}^N)$ and there exists $v \in D^{1,2}(\mathbb{R}^N)$ with $v \neq 0$ such that $v_n \rightharpoonup v$ in $D^{1,2}(\mathbb{R}^N)$. Then, the proof follows from the arguments used in Case 1. \square

From Proposition 1.2, we know that functions of type

$$U_b(\cdot - a) = \frac{C_0(b^2)^{\frac{N-2}{4}}}{(b^2 + |x - a|^2)^{\frac{N-2}{2}}}, \quad \text{for some } C_0, b \in \mathbb{R} \quad \text{and} \quad a \in \mathbb{R}^N$$

achieves the minimum of J_* on \mathcal{M}_* and the minimum value is exactly

$$J_*(U_b(\cdot - a)) = \frac{N+2-\mu}{4N-2\mu} \int_{\mathbb{R}^N} |\nabla U_b(\cdot - a)|^2 dx = \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}} = c_*.$$

Let $\{u_n\} \subset H_0^1(\Omega)$ be a (PS) sequence for J_* at level c_* . Then, Lemma 4.3 implies that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \left| \nabla \left(u_n - \frac{C_0(b_{1,n}^2)^{\frac{N-2}{4}}}{(b_{1,n}^2 + |x - x_{1,n}|^2)^{\frac{N-2}{2}}} \right) \right|^2 dx \rightarrow 0$$

or

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \left| \nabla \left(u_n(x + y_n) - \frac{C_0(b_{2,n}^2)^{\frac{N-2}{4}}}{(b_{2,n}^2 + |x - x_{2,n}|^2)^{\frac{N-2}{2}}} \right) \right|^2 dx \rightarrow 0,$$

which means,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \left| \nabla \left(u_n(x) - \frac{C_0(b_{2,n}^2)^{\frac{N-2}{4}}}{(b_{2,n}^2 + |x - y_n - x_{2,n}|^2)^{\frac{N-2}{2}}} \right) \right|^2 dx \rightarrow 0$$

for some sequence $b_{1,n}, b_{2,n} \in \mathbb{R} \setminus \{0\}$ and $x_{1,n}, x_{2,n} \in \mathbb{R}^N$. Notice that $\text{supp } u_n \in \Omega$, we have $x_{1,n}, y_n + x_{2,n} \in \Omega$, then there exists some sequence $b_n \in \mathbb{R} \setminus \{0\} \rightarrow 0$ and $x_n \in \Omega$ such that

$$(4.9) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla(u_n(x) - U_{b_n}(x - x_n))|^2 dx \rightarrow 0.$$

What's more, we can observe from

$$(4.10) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N \setminus \Omega} \left| \nabla \frac{C_0(b_n^2)^{\frac{N-2}{4}}}{(b_n^2 + |x - x_n|^2)^{\frac{N-2}{2}}} \right|^2 dx \rightarrow 0$$

that $b_n \rightarrow 0$ as n goes to infinity.

Proposition 4.4. *There exists $\beta^* = \beta^*(r) \in (0, \beta_1)$ such that, for $0 < \beta \leq \beta^*$, $\alpha(u) \in \Omega_r^+$ for every $u \in \mathcal{M}_{\beta, \Omega}$ with $J_{\beta, \Omega}(u) \leq c(\beta, B_r)$.*

Proof. As in [37], we argue by contradiction. Assume that there exist sequences $\varepsilon_n \rightarrow 0$, $\beta_n \rightarrow 0$ and $u_n \in \mathcal{M}_{\beta_n, \Omega}$ such that

$$J_{\beta_n, \Omega}(u_n) < c(\beta_n, B_r) + \varepsilon_n \quad \text{and} \quad \alpha(u_n) \notin \Omega_r^+.$$

Then, by Lemma 4.2, we have $J_{\beta_n, \Omega}(u_n) \rightarrow c_*$ and $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Let t_n such that $t_n u_n \in \mathcal{M}_*$. Using Lemma 4.1 and $u_n \in \mathcal{M}_{\beta_n, \Omega}$, we know $t_n \rightarrow 1$. Thanks to $J_{\beta_n, \Omega}(u_n) \rightarrow c_*$, we know

$$\begin{aligned} J_{\beta, \Omega}(u_n) - J_*(t_n u_n) &= \frac{N+2-\mu}{4N-2\mu} (1 - t_n^2) \int_{\mathbb{R}^N} |\nabla u_n|^2 dx - \left(\frac{\beta}{2} - \frac{\beta}{2 \cdot 2_\mu^*} \right) \int_{\mathbb{R}^N} |u_n|^2 dx = o_n(1), \end{aligned}$$

leads to the fact that $J_*(t_n u_n) \rightarrow c_*$. Thus, $\{t_n u_n\}$ is a (PS) sequence for J_* at level c_* . By (4.9), we have

$$t_n u_n - U_{b_n}(\cdot - x_n) \rightarrow 0 \quad \text{in} \quad D^{1,2}(\mathbb{R}^N)$$

for some sequence $b_n \in \mathbb{R} \setminus \{0\}$ and $x_n \in \Omega$. Then, we can write

$$t_n u_n = U_{b_n}(\cdot - x_n) - v_n,$$

where v_n such that $\int_{\mathbb{R}^N} |\nabla v_n|^2 dx \rightarrow 0$ and $U_{b_n}(\cdot - x_n) = v_n$ on $\mathbb{R}^N \setminus \Omega$. We write $x \in \mathbb{R}^N$ as $x = (x_{(1)}, x_{(2)}, \dots, x_{(N)})$, the i -th coordinate of the barycenter of u_n satisfies

$$(4.11) \quad \begin{aligned} \alpha(u_n)_{(i)} \int_{\mathbb{R}^N} |\nabla(t_n u_n)|^2 dx \\ = \int_{\mathbb{R}^N} x_{(i)} |\nabla U_{b_n}(\cdot - x_n)|^2 dx + \int_{\mathbb{R}^N} x_{(i)} |\nabla v_n|^2 dx - 2 \int_{\mathbb{R}^N} x_{(i)} \nabla U_{b_n}(\cdot - x_n) \nabla v_n dx. \end{aligned}$$

Using $U_{b_n}(\cdot - x_n) = v_n$ on $\mathbb{R}^N \setminus \Omega$, we have

$$(4.12) \quad \begin{aligned} \alpha(u_n)_{(i)} \int_{\mathbb{R}^N} |\nabla(t_n u_n)|^2 dx \\ = \int_{\Omega} x_{(i)} |\nabla U_{b_n}(\cdot - x_n)|^2 dx + \int_{\Omega} x_{(i)} |\nabla v_n|^2 dx - 2 \int_{\Omega} x_{(i)} \nabla U_{b_n}(\cdot - x_n) \nabla v_n dx \\ = A_n + B_n - 2D_n. \end{aligned}$$

By simple computations, we know that

$$(4.13) \quad A_n = b_n \int_{\Omega'_n} y_{(i)} |\nabla U_1(y)|^2 dy + (x_n)_{(i)} \int_{\Omega'_n} |\nabla U_1(y)|^2 dy,$$

where $\Omega'_n = \{y \in \mathbb{R}^N : y = x - x_n, x \in \Omega\}$. Since $b_n \rightarrow 0$, we get $b_n \int_{\Omega'_n} y_{(i)} |\nabla U_1(y)|^2 dy = o_n(1)$. From $\int_{\mathbb{R}^N} |\nabla v_n|^2 dx \rightarrow 0$, we get $B_n = o_n(1)$. Since

$$\int_{\Omega} x_{(i)} \nabla U_{b_n}(\cdot - x_n) \nabla v_n dx \leq C \left(\int_{\Omega} |\nabla U_{b_n}(\cdot - x_n)|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |\nabla v_n|^2 dx \right)^{\frac{1}{2}},$$

then, $D_n = o_n(1)$. We know that $\int_{\mathbb{R}^N} |\nabla(t_n u_n)|^2 dx = \int_{\mathbb{R}^N} |\nabla U_1(x)|^2 dx + o_n(1)$. In fact, we have shown that

$$(4.14) \quad \alpha(u_n)_{(i)} = \frac{(x_n)_{(i)} \int_{\Omega'_n} |\nabla U_1(x)|^2 dx + o_n(1)}{\int_{\mathbb{R}^N} |\nabla U_1(x)|^2 dx + o_n(1)}.$$

Since $x_n \in \Omega$ and $\Omega'_n \subset \mathbb{R}^N$, (4.14) implies that $\alpha(u_n) \in \overline{\Omega}$ which is in contrast with assumption and proves the proposition. \square

We choose $R > 0$ such that $\overline{\Omega} \subset B_R$ and set

$$\eta(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq R, \\ R/t & \text{if } R \leq t. \end{cases}$$

On $D^{1,2}(\mathbb{R}^N)$ we define

$$(4.15) \quad \alpha_c(u) = \frac{\int_{\mathbb{R}^N} x \eta(|x|) |\nabla u|^2 dx}{\int_{\mathbb{R}^N} |\nabla u|^2 dx}.$$

Proposition 4.5. *There exist $0 < \beta^* < \beta_1$ and for each $0 < \beta \leq \beta^*$ a number $\lambda_\beta \geq \lambda_\beta$ such that, $\alpha_c(u) \in \Omega_{2r}^+$ for every $\lambda \geq \lambda_\beta$ and $u \in \mathcal{M}_{\lambda,\beta}$ with $J_{\lambda,\beta}(u) \leq c(\beta, B_r)$.*

Proof. Due to the appearance of the convolution part, we adapt the arguments in [13] to suit the new situation. Assume by contradiction that, for $\beta > 0$ arbitrarily small, there is a sequence $\{u_n\} \subset \mathcal{M}_{\lambda_n,\beta}$ such that $\lambda_n \rightarrow \infty$, $J_{\lambda_n,\beta}(u_n) \rightarrow c \leq c(\beta, B_r)$ and $\alpha_c(u_n) \notin \Omega_{2r}^+$. By the proof of Proposition 2.3, we know $\{u_n\}$

is bounded in E . By Lemma 2.1, there is a $v_\beta \in H_0^1(\Omega)$ such that, up to a subsequence, $u_n \rightharpoonup v_\beta$ in E and $u_n \rightarrow v_\beta$ in $L^2(\mathbb{R}^N)$. Next we continue the proof by distinguishing two cases:

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_\beta(x)|^{2^*} |v_\beta(y)|^{2^*}}{|x-y|^\mu} dx dy \leq \langle L_{0,\beta} v_\beta, v_\beta \rangle$$

and

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_\beta(x)|^{2^*} |v_\beta(y)|^{2^*}}{|x-y|^\mu} dx dy > \langle L_{0,\beta} v_\beta, v_\beta \rangle.$$

Case 1. $\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_\beta(x)|^{2^*} |v_\beta(y)|^{2^*}}{|x-y|^\mu} dx dy \leq \langle L_{0,\beta} v_\beta, v_\beta \rangle.$

Since $\{u_n\} \subset \mathcal{M}_{\lambda_n, \beta}$, $\lambda_n \rightarrow \infty$ and $J_{\lambda_n, \beta}(u_n) \rightarrow c \leq c(\beta, B_r) < \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}$. We write $v_n := u_n - v_\beta$. By the proof of the asymptotic behavior of the solutions of (1.8) in Theorem 1.3, we know,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \rightarrow \int_{\mathbb{R}^N} |\nabla v_\beta|^2 dx.$$

Consequently, $\alpha_c(u_n) \rightarrow \alpha(v_\beta)$. However, $J_{\beta, \Omega}(v_\beta) \leq \lim_{n \rightarrow \infty} J_{\lambda_n, \beta}(u_n) \leq c(\beta, B_r)$, it follows from Proposition 4.4 that $\alpha(v_\beta) \in \Omega_r^+$, this is a contradiction.

Case 2. $\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_\beta(x)|^{2^*} |v_\beta(y)|^{2^*}}{|x-y|^\mu} dx dy > \langle L_{0,\beta} v_\beta, v_\beta \rangle.$

By the arguments of Proposition 2.3 and $c(\beta, B_r) < \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}$, we know $\|u_n\|_{L_{\lambda_n, \beta}}$ is bounded uniformly in $0 < \beta < \beta_1$ and $\lambda \geq \lambda_\beta$. Thanks to the proof of Lemma 4.1, we know $|v_\beta|_2^2$ is bounded uniformly in $0 < \beta < \beta_1$. Then, $\beta|v_\beta|_2^2 = o_\beta(1)$ and $\beta|u_n|_2^2 = o_\beta(1)$ for $\beta > 0$ small enough. It is easy to see that there exists $t_\beta \in (0, 1)$ such that $t_\beta v_\beta \in \mathcal{M}_{\beta, \Omega}$. Then, we have

$$t_\beta^2 \int_{\mathbb{R}^N} |\nabla v_\beta|^2 dx = t_\beta^{22^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_\beta(x)|^{2^*} |v_\beta(y)|^{2^*}}{|x-y|^\mu} dx dy + \beta t_\beta^2 \int_{\mathbb{R}^N} |v_\beta|^2 dx.$$

Combining this with the fact that $\{u_n\} \subset \mathcal{M}_{\lambda_n, \beta}$ we get

$$J_{\beta, \Omega}(t_\beta v_\beta) = \frac{N+2-\mu}{4N-2\mu} t_\beta^2 \int_{\mathbb{R}^N} |\nabla v_\beta|^2 dx - \beta \frac{N+2-\mu}{4N-2\mu} t_\beta^2 \int_{\mathbb{R}^N} |v_\beta|^2 dx$$

and

$$J_{\lambda_n, \beta}(u_n) = \frac{N+2-\mu}{4N-2\mu} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + \lambda_n V |u_n|^2) dx - \beta \frac{N+2-\mu}{4N-2\mu} \int_{\mathbb{R}^N} |u_n|^2 dx.$$

Thus,

$$\begin{aligned} c(\beta, \Omega) + \beta \frac{N+2-\mu}{4N-2\mu} t_\beta^2 \int_{\mathbb{R}^N} |v_\beta|^2 dx &\leq \frac{N+2-\mu}{4N-2\mu} t_\beta^2 \int_{\mathbb{R}^N} |\nabla v_\beta|^2 dx \\ &\leq \lim_{n \rightarrow \infty} \frac{N+2-\mu}{4N-2\mu} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \\ &\leq \lim_{n \rightarrow \infty} \frac{N+2-\mu}{4N-2\mu} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + \lambda_n V |u_n|^2) dx \\ &\leq c(\beta, B_r) + \beta \frac{N+2-\mu}{4N-2\mu} \int_{\mathbb{R}^N} |u_n|^2 dx. \end{aligned}$$

It follows that, for n large enough,

$$\left| \int_{\mathbb{R}^N} |\nabla u_n|^2 dx - t_\beta^2 \int_{\mathbb{R}^N} |\nabla v_\beta|^2 dx \right| \leq \frac{4N-2\mu}{N+2-\mu} (c(\beta, B_r) - c(\beta, \Omega)) + o_\beta(1).$$

Since $|c(\beta, B_r) - c(\beta, \Omega)| \rightarrow 0$ as $\beta \rightarrow 0$, this implies that $\left| \int_{\mathbb{R}^N} |\nabla u_n|^2 dx - t_\beta^2 \int_{\mathbb{R}^N} |\nabla v_\beta|^2 dx \right| < r$ for all β sufficiently small. But, by Proposition 4.4, there holds $\alpha(t_\beta v_\beta) \in \Omega_r^+$ which contradicts to the assumption $\alpha_c(u_n) \notin \Omega_{2r}^+$ again. \square

For convenience, we denote $J_{\lambda,\beta}^{\leq c(\beta,B_r)} = \{z \in \mathcal{M}_{\lambda,\beta} : J_{\lambda,\beta}(z) \leq c(\beta, B_r)\}$.

Proof of Theorem 1.4. For $0 < \beta \leq \beta^*$ and $\lambda \geq \lambda_\beta$, we define two maps

$$\Omega_r^- \xrightarrow{\psi_{\beta,r}} J_{\lambda,\beta}^{\leq c(\beta,B_r)} \xrightarrow{h \circ \alpha_c} \Omega_r^-$$

as follow: The map α_c is defined in (4.15) and $h : \Omega_{2r}^+ \rightarrow \Omega_r^-$ is the homotopic equivalence map such that $h|_{\Omega_r^-}$ is the identity. Let $u_{\beta,B_r} \in H_0^1(B_r)$ be a minimizer of $c(\beta, B_r)$ on \mathcal{M}_{β,B_r} . We define the map $\psi_{\beta,r} : \Omega_r^- \rightarrow \mathcal{M}_{\lambda,\beta}$ by

$$\psi_{\beta,r}(y)(x) = \begin{cases} u_{\beta,B_r}(x-y) & \text{if } x \in B_r(y), \\ 0 & \text{if } x \in \Omega \setminus B_r(y). \end{cases}$$

Then, we can see that $\psi_{\beta,r}(y)(x) \equiv 0$ in $\mathbb{R}^N \setminus \Omega$ for every $y \in \Omega_r^-$, it follows that $\alpha_c(\psi_{\beta,r}(y)(x)) \in B_r(y)$, $\psi_{\beta,r}(y)(x) \in \mathcal{M}_{\lambda,\beta}$ and $J_{\lambda,\beta}(\psi_{\beta,r}(y)(x)) = J_{\beta,B_r}(\psi_{\beta,r}(y)(x)) = c(\beta, B_r)$. Thus $\psi_{\beta,r}$ is also well defined. Moreover, $\alpha_c \circ \psi_{\beta,r}$ is the inclusion $\Omega_r^- \rightarrow \Omega_{2r}^+$. Then we know the composite map $h \circ \alpha_c \circ \psi_{\beta,r}$ is homotopic to the identity of Ω_r^- . By a property of the category, we get

$$\text{cat}_{J_{\lambda,\beta}^{\leq c(\beta,B_r)}}(J_{\lambda,\beta}^{\leq c(\beta,B_r)}) \geq \text{cat}_{\Omega_r^-}(\Omega_r^-)$$

(see e.g. [25]) and the choice of r gives $\text{cat}_{\Omega_r^-}(\Omega_r^-) = \text{cat}_{\overline{\Omega}}(\overline{\Omega})$. It follows from Proposition 2.3 that the (PS) condition is verified on $\mathcal{M}_{\lambda,\beta}$, by applying the Lusternik-Schnirelmann theory (see e.g. [34, 41]) we obtain the existence of at least $\text{cat}_{\overline{\Omega}}(\overline{\Omega})$ critical points for $J_{\lambda,\beta}$ on the manifold $\mathcal{M}_{\lambda,\beta}$ which are the solutions of (1.8). The proof is completed. \square

5. EXISTENCE OF SOLUTIONS FOR THE CASE $\beta > \beta_1$

In the following we consider the critical Choquard equation (1.8) with indefinite potential. Assume that, $0 < \mu < 4$, $N \geq 4$, $\beta > \beta_1$, $\beta \neq \beta_j$ for any $j > 1$ and the potential $V(x)$ satisfies (V_1) and (V_3) .

As above sections, we still denote the operator $L_{\lambda,\beta} := -\Delta + \lambda V(x) - \beta$, particularly, $L_{0,\beta} = -\Delta - \beta$. In the following we denote by $|L_{\lambda,\beta}|$ the absolute value of operator $L_{\lambda,\beta}$ and let $E_\lambda = D(|L_{\lambda,\beta}|^{\frac{1}{2}})$ be the Hilbert space equipped with the inner product

$$(u_1, u_2) = \langle |L_{\lambda,\beta}|^{\frac{1}{2}} u_1, |L_{\lambda,\beta}|^{\frac{1}{2}} u_2 \rangle$$

and the norm

$$\|u\|_{L_{\lambda,\beta}} = (u, u)^{\frac{1}{2}}.$$

By conditions (V_1) and (V_3) , E_λ is continuously embedded in $H^1(\mathbb{R}^N)$ for λ large enough.

By condition (V_1) and Remark 1.6, we know that the zero set of $V(x)$ is a bounded domain in \mathbb{R}^N and so we have that $\inf \sigma_e(L_{\lambda,\beta}) \geq \lambda M_0$ and $L_{\lambda,\beta}$ has finite Morse index on E_λ , where $\sigma_e(L_{\lambda,\beta})$ denote the essential spectrum of operator $L_{\lambda,\beta}$ in E_λ and M_0 is the same constant appeared in Remark 1.6. Thus E_λ splits as an orthogonal sum $E_\lambda = E_\lambda^- \oplus E_\lambda^0 \oplus E_\lambda^+$ according to the negative, zero and positive eigenspace of $L_{\lambda,\beta}$ and $\dim E_\lambda^- \cup E_\lambda^0 < \infty$. On the other hand, since $\inf \sigma_e(L_{\lambda,\beta}) \geq \lambda M_0$, we may assume that $\zeta_1^\lambda < \zeta_2^\lambda < \dots < \zeta_{k_\lambda}^\lambda < e(L_{\lambda,\beta})$ be the distinct eigenvalues of $L_{\lambda,\beta}$ in E_λ and $k_\lambda \in \mathbb{N}$ goes to ∞ as $\lambda \rightarrow \infty$. The operator $L_{0,\beta}$ has discrete spectrum in $H_0^1(\Omega)$ and we denote them as

$$\zeta_1 < \zeta_2 < \dots < \zeta_j < \zeta_{j+1} < \dots, \quad \zeta_j = \beta_j - \beta$$

which are the distinct eigenvalues of $L_{0,\beta}$ in $H_0^1(\Omega)$. Let $\mathbb{F}_j^\lambda (j \leq k_\lambda)$ be the corresponding eigenspaces of ζ_j^λ and \mathbb{F}_j be the corresponding eigenspaces of ζ_j . Involving the relationship the eigenspaces, the following two Lemmas are taken from [39].

Lemma 5.1. $\zeta_j^\lambda \rightarrow \zeta_j$ and $\mathbb{F}_j^\lambda \rightarrow \mathbb{F}_j$ as $\lambda \rightarrow \infty$.

Here $\mathbb{F}_j^\lambda \rightarrow \mathbb{F}_j$ means that, given any sequence $\lambda_i \rightarrow \infty$ and normalized eigenfunctions $\varphi_i \in \mathbb{F}_j^{\lambda_i}$, there exists a normalized eigenfunction $\varphi \in \mathbb{F}_j$ such that $\varphi_i \rightarrow \varphi$ strongly in $H^1(\mathbb{R}^N)$ along a subsequence.

Lemma 5.2. *For λ large the operator $L_{\lambda,\beta}$ on E_λ is non-degenerate and has finite Morse index uniformly in λ .*

By Lemma 5.2, we can see that for λ large, E_λ^0 is indeed the zero space $\{0\}$, which implies that for λ large, we have $E_\lambda = E_\lambda^- \oplus E_\lambda^+$. So, we have

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + (\lambda V(x) - \beta)u^2)dx = \|u^+\|_{L_{\lambda,\beta}}^2 - \|u^-\|_{L_{\lambda,\beta}}^2$$

and

$$J_{\lambda,\beta}(u) = \frac{1}{2}\|u^+\|_{L_{\lambda,\beta}}^2 - \frac{1}{2}\|u^-\|_{L_{\lambda,\beta}}^2 - \frac{1}{2 \cdot 2_\mu^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2_\mu^*} |u(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy,$$

where $u = u^+ + u^- \in E_\lambda^+ \oplus E_\lambda^-$. We define the corresponding Nehari manifold as follows:

$$\mathcal{N}_\lambda := \{u \in E_\lambda \setminus \{0\} : \langle J'_{\lambda,\beta}(u), u \rangle = 0\}.$$

and denote

$$(5.1) \quad c_\lambda := \inf_{u \in \mathcal{N}_\lambda} J_{\lambda,\beta}(u).$$

In next section, we will show that for λ large, (1.8) admits a ground state solutions u_λ which achieves c_λ for $\lambda > 0$ large such that u_λ converge as $\lambda \rightarrow \infty$ towards a ground state solution of (1.10) that lies on the level

$$(5.2) \quad c(\beta, \Omega) := \inf_{u \in \mathcal{N}_{\beta,\Omega}} J_{\beta,\Omega}(u).$$

where $\mathcal{N}_{\beta,\Omega} := \{u \in H_0^1(\Omega) \setminus \{0\} : \langle J'_{\beta,\Omega}(u), u \rangle = 0\}$ and $J_{\beta,\Omega}$ is the corresponding variational functional of (1.10), see Section 3.

For $r > 0$, we set $B_r^+ = \{u \in E_\lambda^+ : \|u\|_{L_{\lambda,\beta}} \leq r\}$ and $S_r^+ = \{u \in E_\lambda^+ : \|u\|_{L_{\lambda,\beta}} = r\}$, and for $w \in E_\lambda^+$, we define the convex subset

$$H_w := \{v + tw : v \in E_\lambda^-, t \geq 0\} \subset E_\lambda.$$

Lemma 5.3. *The functional $J_{\lambda,\beta}$ satisfies the following conditions:*

- (i) *There exist $r, \alpha > 0$ such that $J_{\lambda,\beta}|_{S_r^+}(u) \geq \alpha$ and $J_{\lambda,\beta}|_{B_r^+}(u) \geq 0$.*
- (ii) *For any $w \in E_\lambda^+ \setminus \{0\}$, there exists $R_w > 0$ and $C_w > 0$ such that $J_{\lambda,\beta}(u) < 0$ for all $u \in H_w \setminus B_{R_w}$ and $\max_{u \in H_w} J_{\lambda,\beta}(u) \leq C_w$.*

Proof. (i) By the Sobolev embedding and Hardy-Littlewood-Sobolev inequality, for all $u \in E_\lambda^+ \setminus \{0\}$ we have

$$\begin{aligned} J_{\lambda,\beta}(u) &= \frac{1}{2}\|u\|_{L_{\lambda,\beta}}^2 - \frac{1}{2 \cdot 2_\mu^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2_\mu^*} |u(y)|^{2_\mu^*}}{|x-y|^\mu} dxdy \\ &\geq \frac{1}{2}\|u\|_{L_{\lambda,\beta}}^2 - \frac{1}{2 \cdot 2_\mu^*} C_1 |u|_{2^*}^{2 \cdot 2_\mu^*} \\ &\geq \frac{1}{2}\|u\|_{L_{\lambda,\beta}}^2 - C_2 \|u\|_{L_{\lambda,\beta}}^{2 \cdot 2_\mu^*}. \end{aligned}$$

Since $2 < 2 \cdot 2_\mu^*$, we can choose some $r, \alpha > 0$ such that $J_{\lambda,\beta}|_{S_r^+}(u) \geq \alpha$ and $J_{\lambda,\beta}|_{B_r^+}(u) \geq 0$.

(ii) We only need to show if $\mathcal{V} \subset E_\lambda^+ \setminus \{0\}$ is a compact subset, then there exists $R > 0$ such that $J_{\lambda,\beta} < 0$ on $H_w \setminus B_R$ for every $w \in \mathcal{V}$.

As in [38], we may assume that $\|w\|_{L_{\lambda,\beta}} = 1$ for every $w \in \mathcal{V}$. Suppose by contradiction that there exist $w_n \in \mathcal{V}$ and $u_n \in H_{w_n}$, $n \in \mathbb{N}$, such that $J_{\lambda,\beta}(u_n) \geq 0$ for all n and $\|u_n\|_{L_{\lambda,\beta}} \rightarrow \infty$ as $n \rightarrow \infty$. Passing to a subsequence, we may assume that $w_n \rightarrow w_0 \in E_\lambda^+$, $\|w_0\|_{L_{\lambda,\beta}} = 1$. Set $v_n = \frac{u_n}{\|u_n\|_{L_{\lambda,\beta}}} = t_n w_n + v_n^-$, then

$$(5.3) \quad 0 \leq \frac{J_{\lambda,\beta}(u_n)}{\|u_n\|_{L_{\lambda,\beta}}^2} = \frac{1}{2}(t_n^2 - \|v_n^-\|_{L_{\lambda,\beta}}^2) - \frac{1}{2 \cdot 2_\mu^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2_\mu^*-1} |v_n(x)| |u_n(y)|^{2_\mu^*-1} |v_n(y)|}{|x-y|^\mu} dxdy.$$

Hence $\|v_n^-\|_{L_{\lambda,\beta}}^2 \leq t_n^2 = 1 - \|v_n^-\|_{L_{\lambda,\beta}}^2$ and $\frac{1}{\sqrt{2}} \leq t_n \leq 1$. So, for a subsequence, $t_n \rightarrow t_0 > 0$, $v_n \rightharpoonup v_0$ in E_λ and $v_n(x) \rightarrow v_0(x)$ a.e. in \mathbb{R}^N . Hence $v_0 = t_0 w_0 + v_0^- \neq 0$ and, since $|u_n(x)| \rightarrow \infty$ if $v_0(x) \neq 0$,

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2^*_\mu-1} |v_n(x)| |u_n(y)|^{2^*_\mu-1} |v_n(y)|}{|x-y|^\mu} dx dy \rightarrow \infty,$$

contrary to (5.3). \square

Define

$$c^* := \inf_{w \in E_\lambda^+ \setminus \{0\}} \max_{u \in H_w} J_{\lambda,\beta}(u).$$

As a consequence of Lemma 5.3 we have

Corollary 5.4. *There exist $\alpha, C > 0$ such that $\alpha \leq c^* < C$.*

Following Ackermann [2], for a fixed $u \in E_\lambda^+$ we introduce $\Phi_u : E_\lambda^- \rightarrow \mathbb{R}$ defined by

$$\Phi_u(v) = J_{\lambda,\beta}(u+v).$$

Let $\Psi(u) := \frac{1}{2 \cdot 2_\mu^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2_\mu^*} |u(y)|^{2_\mu^*}}{|x-y|^\mu} dx dy$, by direct computation and $\mu < 4$, we know

$$\langle \Psi''(u)w, w \rangle \geq 0$$

for all $u, w \in E_\lambda$, and hence

$$\langle \Phi_u''(v)w, w \rangle = \langle J_{\lambda,\beta}''(u+v)w, w \rangle = -\|w\|_{L_{\lambda,\beta}}^2 - \langle \Psi''(u+v)w, w \rangle \leq -\|w\|_{L_{\lambda,\beta}}^2.$$

In addition,

$$\Phi_u(v) \leq \frac{1}{2} \|u\|_{L_{\lambda,\beta}}^2 - \frac{1}{2} \|v\|_{L_{\lambda,\beta}}^2.$$

Therefore Φ_u is strictly concave and $\lim_{\|v\|_{L_{\lambda,\beta}} \rightarrow \infty} \Phi_u(v) = -\infty$. From weak sequential upper semicontinuity of Φ_u , it follows that there is a unique strict maximum point $h(u) \in E_\lambda^-$ for Φ_u , which is also the only critical point of Φ_u on E_λ^- . Thus $h(u)$ satisfies

$$(5.4) \quad \langle \Phi_u'(h(u)), v \rangle = 0$$

for all $v \in E_\lambda^-$, and

$$v \neq h(u) \Leftrightarrow J_{\lambda,\beta}(u+v) < J_{\lambda,\beta}(u+h(u)).$$

As [2], Lemma 5.6], we have the following:

- Lemma 5.5.** (i) h is \mathbb{R}^N -invariant, i.e. $h(a * u) = h(u)$ where $(a * u)(x) := u(x+a)$ for all $a \in \mathbb{R}^N$.
(ii) $h \in C^1(E_\lambda^+, E_\lambda^-)$ and $h(0) = 0$.
(iii) h is a bounded map.
(iv) If $u_n \rightharpoonup u$ in E_λ^+ , then $h(u_n) - h(u_n - u) \rightarrow h(u)$ and $h(u_n) \rightharpoonup h(u)$. The same is true for $|h(u)|_2^2$.

Define $\Upsilon : E_\lambda^+ \rightarrow \mathbb{R}$ by

$$\Upsilon(u) = J_{\lambda,\beta}(u+h(u)) = \frac{1}{2} \|u\|_{L_{\lambda,\beta}}^2 - \frac{1}{2} \|h(u)\|_{L_{\lambda,\beta}}^2 - \Psi(u+h(u)).$$

By Theorem 5.1 in [2], we know that the critical points of Υ and $J_{\lambda,\beta}$ are one to one correspondence via the injective map $u \rightarrow u+h(u)$ from E_λ^+ into E_λ .

Let

$$\mathcal{N} := \{u \in E_\lambda^+ \setminus \{0\} : \langle \Upsilon'(u), u \rangle = 0\},$$

and we define

$$c^{**} = \inf_{u \in \mathcal{N}} \Upsilon(u).$$

Lemma 5.6. $c^* = c^{**} = c_\lambda$, where c_λ is defined in (5.1).

Proof. As in [16], given $w \in E_\lambda^+$, if $u = tw + v \in H_w$ with $J_{\lambda,\beta}(u) = \max_{z \in H_w} J_{\lambda,\beta}(z)$ then the restriction $J_{\lambda,\beta}|_{H_w}$ of $J_{\lambda,\beta}$ on H_w satisfies $(J_{\lambda,\beta}|_{H_w})'(u) = 0$ which implies $v = h(tw)$ and $\langle \Upsilon'(tw), tw \rangle = \langle J'_{\lambda,\beta}(u), tw \rangle = 0$, i.e. $tw \in \mathcal{N}$. Thus $c^* \geq c^{**}$. On the other hand, if $e \in \mathcal{N}$ then $(J_{\lambda,\beta}|_{H_e})'(e + h(e)) = 0$ so $c^* \leq \max_{z \in H_e} J_{\lambda,\beta}(z) = \Upsilon(e)$. Thus $c^* \leq c^{**}$ and similarly, $c_\lambda \leq c^{**}$. This proves $c^* = c^{**}$.

For any $w \in \mathcal{N}_\lambda$, we have $w^+ + w^- \in E_\lambda^+ \oplus E_\lambda^-$ and $w^+ \neq 0$. So $w \in H_{w^+}$. Combining this with the fact that $\langle J'_{\lambda,\beta}(w), w \rangle = 0$ and the discussion before Lemma 5.5, we have

$$J_{\lambda,\beta}(w) = \max_{u \in H_{w^+}} J_{\lambda,\beta}(u),$$

that is $J_{\lambda,\beta}(w) \geq c^*$ and so $c_\lambda \geq c^*$. Together with the fact that $c_\lambda \leq c^{**}$, we have $c_\lambda = c^{**}$. This proves $c_\lambda = c^{**} = c^*$. \square

6. THE PROOF OF THEOREM 1.5

Now, we prove that for λ large enough, any Palais-Smale sequence is bounded. For this, we define

$$X_{NL} := \{u : \mathbb{R}^N \rightarrow \mathbb{R}; \|u\|_{NL} < +\infty\},$$

where

$$\|\cdot\|_{NL} := \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\cdot|^{2_\mu^*} |\cdot|^{2_\mu^*}}{|x-y|^\mu} dx dy \right)^{\frac{1}{2 \cdot 2_\mu^*}}.$$

By Lemma 2.3 of [18], we know $\|\cdot\|_{NL}$ defines a norm on X_{NL} under which X_{NL} is a Banach space. Moreover the Hardy-Littlewood-Sobolev inequality also implies that $H^1(\mathbb{R}^N)$ is continuously embedded in X_{NL} .

Lemma 6.1. *If $\{u_n\}$ is a $(PS)_{c_\lambda}$ sequence for $J_{\lambda,\beta}$, then $\{u_n\}$ is bounded in E_λ .*

Proof. Let $\vartheta \in (\frac{1}{2 \cdot 2_\mu^*}, \frac{1}{2})$. It follows from $\{u_n\}$ is a $(PS)_{c_\lambda}$ sequence that, for n large enough, we have

$$\begin{aligned} c_\lambda + o_n(1) \|u_n\|_{L_{\lambda,\beta}} &\geq J_{\lambda,\beta}(u_n) - \vartheta \langle J'_{\lambda,\beta}(u_n), u_n \rangle \\ &= \left(\frac{1}{2} - \vartheta\right) \int_{\mathbb{R}^N} (|\nabla u_n|^2 + (\lambda V(x) - \beta)|u_n|^2) dx + \left(\vartheta - \frac{1}{2 \cdot 2_\mu^*}\right) \|u_n\|_{NL}^{2 \cdot 2_\mu^*} \\ &= \left(\frac{1}{2} - \vartheta\right) (\|u_n^+\|_{L_{\lambda,\beta}}^2 - \|u_n^-\|_{L_{\lambda,\beta}}^2) + \left(\vartheta - \frac{1}{2 \cdot 2_\mu^*}\right) \|u_n\|_{NL}^{2 \cdot 2_\mu^*}, \end{aligned}$$

where $u_n = u_n^+ + u_n^- \in E_\lambda^+ \oplus E_\lambda^-$. It is then easy to verify that $\{u_n\}$ is bounded in E_λ by using the fact that that E_λ^- is finite dimensional and $\|\cdot\|_{NL}$ is a norm in X_{NL} . This completes the proof of Lemma 6.1. \square

Enlarging λ_β if necessary, we may assume that $\lambda_\beta \geq \beta/M_0$, thus

$$\lambda M_0 - \beta \geq 0 \quad \text{for all } \lambda \geq \lambda_\beta,$$

where M_0 is given in Remark 1.6.

Proposition 6.2. *Suppose $\lambda \geq \lambda_\beta$ and $\{u_n\}$ is $(PS)_{c_\lambda}$ sequence of $J_{\lambda,\beta}$ with*

$$c_\lambda < \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}.$$

Then there exists a subsequence of $\{u_n\}$ which converge strongly in E_λ a solution u_λ of (1.8) such that $J_{\lambda,\beta}(u_\lambda) = c_\lambda$.

Proof. By Lemma 6.1, we know that $\{u_n\}$ is bounded in E_λ . Similar to the proof of Proposition 2.3, we can obtain that there exists a subsequence of $\{u_n\}$ which converge strongly in E_λ a solution u_λ of (1.8) such that $J_{\lambda,\beta}(u_\lambda) = c_\lambda$. \square

Lemma 6.3. *For $\lambda > \lambda_\beta$, we have*

$$c_\lambda < \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}.$$

Proof. By the definition of c_λ we know that $c_\lambda \leq c(\beta, \Omega)$, where $c(\beta, \Omega)$ is defined as in (5.2). By Proposition 3.3, we know that

$$c_\lambda < \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}$$

and we complete the proof. \square

Proposition 6.4. *For $\lambda > \lambda_\beta$, there is a ground state solution u_λ of (1.8) which achieves c_λ .*

Proof. Let $\{w_n\} \subset \mathcal{N}$ be a minimization sequence: $\Upsilon(w_n) \rightarrow c^{**}$. By the Ekeland variational principle we can assume that $\{w_n\}$ is, in addition, a $(PS)_{c^{**}}$ sequence for Υ on \mathcal{N} . A standard argument shows that $\{w_n\}$ is in fact a $(PS)_{c^{**}}$ sequence for Υ on E_λ^+ (see, e.g., [41]). Then $\{u_n = w_n + h(w_n)\}$ is a $(PS)_{c_\lambda}$ sequence for $J_{\lambda,\beta}$ on E_λ . By Proposition 6.2 and Lemma 6.3, we have that there is a ground state solution u_λ of (1.8) which achieves c_λ . \square

In the following, we come to give the asymptotic behavior of the ground state solutions of (1.8) as λ goes to infinity.

Proposition 6.5. *$\lim_{\lambda \rightarrow +\infty} c_\lambda = c(\beta, \Omega)$ and for any sequence $\{\lambda_n\}(\lambda_n \rightarrow +\infty)$, up to a subsequence $u_{\lambda_n} \rightarrow u$ strongly in $H^1(\mathbb{R}^N)$. Here u is a ground state solution of (1.10) which achieves $c(\beta, \Omega)$.*

Proof. For $u \in H_0^1(\Omega)$, we have $\mathcal{N}_{\beta,\Omega} \subset \mathcal{N}_\lambda$. Thus by the definition of c_λ and $c(\beta, \Omega)$, it is easy to see that $c_\lambda \leq c(\beta, \Omega)$ for $\lambda \geq \lambda_\beta$. On the other hand, it is not difficult to check that c_λ is nondecreasing as λ growth. Thus we may assume that $\lim_{\lambda \rightarrow +\infty} c_\lambda = \kappa \leq c(\beta, \Omega)$ which implies for any sequence $\{\lambda_n\}(\lambda_n \rightarrow +\infty)$, $c_{\lambda_n} \rightarrow \kappa \leq c(\beta, \Omega)$. We assume that u_n is such that c_{λ_n} is achieved, by Lemma 6.1, $\{u_n\}$ is bounded in E_{λ_n} and thus is also bounded in $H^1(\mathbb{R}^N)$. As a result, we have

$$u_n \rightharpoonup u \text{ weakly in } H^1(\mathbb{R}^N),$$

$$u_n \rightarrow u \text{ strongly in } L_{loc}^q(\mathbb{R}^N) \text{ for } 2 \leq q < 2^*$$

and

$$u_n \rightarrow u \text{ a.e. in } \mathbb{R}^N.$$

We claim that $u|_{\Omega^c} = 0$, where $\Omega^c := \{x | x \in \mathbb{R}^N \setminus \Omega\}$. Indeed, if not, there exists a compact subset $D_1 \subset \Omega^c$ with $\text{dist}\{D_1, \partial\Omega\} > 0$ such that $u|_{D_1} \neq 0$ and

$$\int_{D_1} u_n^2 dx \rightarrow \int_{D_1} u^2 dx > 0.$$

Moreover, there exists $\epsilon_0 > 0$ such that $V(x) \geq \epsilon_0$ for any $x \in D_1$.

By the choice of $\{u_n\}$, we have

$$0 = \langle J'_{\lambda,\beta}(u_n), u_n \rangle = \int_{\mathbb{R}^N} (|\nabla u_n|^2 + (\lambda_n V - \beta)u_n^2) dx - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2_\mu^*} |u_n(y)|^{2_\mu^*}}{|x-y|^\mu} dx dy,$$

hence for n large

$$\begin{aligned} J_{\lambda,\beta}(u_n) &= \left(\frac{1}{2} - \frac{1}{2 \cdot 2_\mu^*}\right) \int_{\mathbb{R}^N} |\nabla u_n|^2 + (\lambda_n V(x) - \beta)u_n^2 dx \\ &\geq \left(\frac{1}{2} - \frac{1}{2 \cdot 2_\mu^*}\right) \int_{D_1} (\lambda_n \epsilon_0 - \beta)u_n^2 dx \rightarrow +\infty \end{aligned}$$

as $n \rightarrow \infty$. This contradiction shows that $u|_{\Omega^c} = 0$. By the smooth assumption on the boundary $\partial\Omega$ we indeed have $u \in H_0^1(\Omega)$.

Now we prove that

$$(6.1) \quad u_n \rightarrow u \text{ strongly in } X_{NL}.$$

We take $v_n := u_n - u$ and suppose on the contrary that (6.1) is not true, then up to a subsequence, we may assume that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n(x)|^{2^*} |v_n(y)|^{2^*}}{|x-y|^\mu} dx dy \rightarrow b > 0.$$

By a similar argument as the proof of Proposition 2.3, we can show that $b \geq S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}$ which implies that $\kappa = \lim_{n \rightarrow \infty} c_{\lambda_n} \geq \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}$. This contradicts with $\kappa < c(\beta, \Omega) < \frac{N+2-\mu}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}$. Namely we proved that (6.1) holds.

From the fact that u_n is the solutions of (1.8) with λ replaced by λ_n , we have

$$\int_{\mathbb{R}^N} (\nabla u_n \nabla \varphi + (\lambda_n V - \beta) u_n \varphi) dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2^*} |u_n(y)|^{2^*} u_n(y) \varphi(y)}{|x-y|^\mu} dx dy$$

for any $\varphi \in E$. If $\varphi \in H_0^1(\Omega)$ then $\lambda_n \int_{\mathbb{R}^N} V u_n \varphi dx = 0$ for all n . Letting $n \rightarrow \infty$ we obtain

$$\int_{\mathbb{R}^N} (\nabla u \nabla \varphi - \beta u \varphi) dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2^*} |u(y)|^{2^*} u(y) \varphi(y)}{|x-y|^\mu} dx dy$$

for any $\varphi \in H_0^1(\Omega)$. So, u is a solution of (1.10). Since $V(x) = 0$ for $x \in \Omega$, we get

$$(6.2) \quad \langle L_{\lambda_n, \beta} u_n, u_n \rangle = \langle L_{0, \beta} u, u \rangle + \langle L_{\lambda_n, \beta} v_n, v_n \rangle,$$

where $v_n = u_n - u$. Since $\{u_n\}$ is a sequence of solutions of (1.8) and u is a solution of (1.10), by (6.1) we can get

$$(6.3) \quad \langle L_{\lambda_n, \beta} v_n, v_n \rangle = o_n(1).$$

Thus, from (6.2) we get

$$\langle L_{\lambda_n, \beta} u_n, u_n \rangle \rightarrow \langle L_{0, \beta} u, u \rangle$$

as $n \rightarrow \infty$, that is

$$\int_{\mathbb{R}^N} (|\nabla u_n|^2 + (\lambda_n V(x) - \beta) u_n^2) dx \rightarrow \int_{\mathbb{R}^N} (|\nabla u|^2 - \beta u^2) dx$$

as $n \rightarrow \infty$. By Lemma 2.1, we know $u_n \rightarrow u$ in $L^2(\mathbb{R}^N)$ and so

$$\int_{\mathbb{R}^N} (|\nabla u_n|^2 + \lambda_n V(x) u_n^2) dx \rightarrow \int_{\mathbb{R}^N} |\nabla u|^2 dx$$

as $n \rightarrow \infty$. It follows from

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \leq \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx$$

that

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \rightarrow \int_{\mathbb{R}^N} |\nabla u|^2 dx$$

as $n \rightarrow \infty$. Combining this with the fact that $u_n \rightarrow u$ in $L^2(\mathbb{R}^N)$, we have

$$u_n \rightarrow u \text{ strongly in } H^1(\mathbb{R}^N).$$

By the definition of $c(\beta, \Omega)$ we have $J_{\beta, \Omega}(u) \geq c(\beta, \Omega)$. On the other hand, by the strong convergence of u_n , we have

$$\begin{aligned} J_{\beta, \Omega}(u) &= \left(\frac{1}{2} - \frac{1}{2 \cdot 2_\mu^*}\right) \int_{\Omega} \int_{\Omega} \frac{|u(x)|^{2^*} |u(y)|^{2^*}}{|x-y|^\mu} dx dy \\ &= \left(\frac{1}{2} - \frac{1}{2 \cdot 2_\mu^*}\right) \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2^*} |u_n(y)|^{2^*}}{|x-y|^\mu} dx dy \\ &= \lim_{n \rightarrow \infty} J_{\lambda, \beta}(u_n) \\ &= \lim_{n \rightarrow \infty} c_{\lambda_n} = \kappa \leq c(\beta, \Omega). \end{aligned}$$

Thus we proved that $J_{\beta,\Omega}(u) = \kappa = c(\beta, \Omega)$. Namely u is indeed a ground state solution of (1.10) which achieves $c(\beta, \Omega)$ and thus the proof of Proposition 6.5 is completed. \square

Proof of Theorem 1.5. This is a direct results of Proposition 6.4 and Proposition 6.5. \square

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